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Mark Burgin

Hypernumbers and Extrafunctions

Extending the Classical Calculus

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Mark Burgin
Department of Mathematics
University of California
Los Angeles, CA 90095, USA

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Preface

Encountering problems that they were not able to solve, creative mathematicians, as a rule, further developed mathematics, finding ways to solve previously "unsolvable" problems. In many cases, such problems and the corresponding development of mathematics have been inspired by science. In such a way, the stimulus to develop the theory of hypernumbers and extrafunctions comes from physics. Exploring mysteries of the microworld, physicists often encounter situations when their formulas acquire infinite values due to divergence of the used series and integrals. At the same time, all measured values are naturally finite. A common way to eliminate this discrepancy between theory and experiment is seemingly artificial manipulation with formulas, for the sole purpose of getting rid of infinite values. Another natural way to deal with such situation is to learn how to rigorously work with infinities and come to finite values given by measurements. Mathematicians suggested several approaches to a rigorous operation with infinite values by introducing infinite numbers. The most popular of them are transfinite numbers of Cantor, nonstandard analysis of Robinson, and surreal numbers. However, all these constructions, which contributed a lot to the development of mathematics, for example, Cantor's set theory, brought very little to the realm of physics.

Contrary to this, there is a mathematical theory called the theory of distributions, which allowed physicists to rigorously operate with functions that take infinite values. An example of such a function is the Heaviside–Dirac delta function. Although at first physicists were skeptical about this theory, now it has become one of the most efficient tools of theoretical physics.

This book is designed to introduce the reader to another rigorous mathematical approach to operation with infinite values. It is called the theory of hypernumbers and extrafunctions. In it, the concepts of real and complex numbers are extended in such a way that the new universe of numbers called hypernumbers includes infinite quantities. It is necessary to remark that in contrast to nonstandard analysis, there are no infinitely small hypernumbers. This is more relevant to the situation in physics, where infinitely big values emerge from theoretical structures but physicists have never encountered infinitely small values. The next step of extending the classical calculus based on real and complex functions is introduction of extrafunctions, which generalize not only the concept of a conventional function but also the concept of a distribution. This made possible to solve previously "unsolvable" problems. For instance, there are linear partial differential equations for which it is proved that they do not have solutions not only in conventional functions but even in distributions. At the same time, it was also proved that all these and many other equations have solutions in extrafunctions, which are studied in this book. Besides, extrafunctions have been efficiently used for a rigorous mathematical definition of the Feynman path integral, as well as for solving some problems in probability theory, which is also important for contemporary physics.

In different papers, the theory of hypernumbers and extrafunctions has been developed in the context of complex numbers, finite and infinite dimensional vector spaces, such as Banach spaces, and abstract spaces with a measure. The aim of this book is not to achieve the highest possible generality or to give an exposition of all known or basic results from the theory of hypernumbers and extrafunctions. Our aspirations are more modest. That is why we restrict our exposition only to real numbers and real functions striving to achieve better understanding of the main ideas, constructions and results obtained in this theory. We want to show that even in the most standard case of real analysis, hypernumbers and extrafunctions significantly extend the scope and increase the power not only of the classical calculus but also of its modern generalizations and extensions, such as distribution theory or gauge integration.

It is necessary to remark that some people erroneously perceive the theory of hypernumbers and extrafunctions as a new version of nonstandard analysis. This is incorrect because these theories are essentially different. First, nonstandard analysis has both infinitely big and infinitely small numbers, while the theory of hypernumbers and extrafunctions has only infinitely big numbers. Second, nonstandard analysis is oriented on problems of calculus foundations, while the theory of hypernumbers and extrafunctions is oriented on problems of physics. Indeed, quantum theory regularly encounters infinitely big values in the form of divergent series and integrals but never meets infinitely small numbers. Third, nonstandard analysis is constructed based on set-theoretical principles, while the theory of hypernumbers and extrafunctions is constructed based on topological principles. Similar distinctions exist between real hypernumbers and surreal numbers, which include hyperreal numbers from nonstandard analysis.

The book may be used for enhancing traditional courses of calculus for undergraduates, as well as for teaching a separate course for graduate students.

Los Angeles, CA, USA Mark Burgin

Contents

Chapter 1
Introduction: How Mathematicians Solve "Unsolvable" Problems

The function of the imagination is not to make strange things settled, so much as to make settled things strange.

(G.K. Chesterton 1874–1936)

Does not it seem like a contradiction or a paradox to *solve an unsolvable problem*? Yet the most courageous and creative mathematicians and scientists are the ones trying to make sense not only of vague ideas but also of paradoxes and contradictions. So let us look how they do this, using as an example the history of the number system development. Analyzing the evolution of numbers, we can see that it has been a process of deficiency elimination. Each step of this process caused absence of understanding, an open opposition, and even hostility to new numbers.

When mathematics was in its earliest stage of development, natural numbers emerged, plausibly, in a natural way from counting. Now natural numbers constitute one of the most basic structures in the whole mathematics. Their importance is vividly expressed by the prominent mathematician Leopold Kronecker, who said: "God made the natural numbers, all the rest is the work of man."

At first, people used only finite sets of natural numbers. This hypothesis is supported by anthropological studies of primitive tribes, which have very small number systems, such as {*one*, *two*, *many*}. At the same time, even developed ancient civilizations, such as Egyptian or Babylonian civilizations, utilized only finite sets of natural numbers as their number representation systems, called numeral systems, had means for representing big but bounded collections of numbers.

However, any finite number system for counting has the following deficiency. If for example, its largest number is n, then it is always possible that a necessity will emerge to count more. To eliminate this deficiency, people made an intellectual effort and leaped from finite number systems to the infinite system of all natural numbers.

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Natural numbers represented many different things. In an extreme form, this is expressed by the motto of Pythagoreans: "Everything is a number." Such prominent mathematician as Emil Borel wrote that all of mathematics can be deduced from the sole notion of an integer number.

Using numbers, people introduced operations with numbers: addition, subtraction, multiplication, and division. This introduction transformed sets of numbers into arithmetic. However, while addition and multiplication were defined for all pairs of natural numbers, it was not so for the two other operations.

When this deficiency was discovered, a need to perform division with all numbers had emerged because people always divided different things, e.g., their crop or other products. As a result, fractions and later rational numbers were introduced, making division also a total operation.

Impossibility to perform subtraction with all pairs of numbers is equivalent to unsolvability of equations of the form $a - b = x$, e.g., $3 - 5 = x$. As a result, zero and then negative numbers were invented. This gave birth to the system of the integer numbers. In such a way, subtraction also became a total operation and one more "unsolvable" problem was solved.

When people began to solve arithmetical and algebraic equations, new deficiencies emerged in the form of the impossibility to solve such simple equations as $x^2 = 2$ and similar ones. To deal with this inconvenient situation, mathematicians used formal expressions like $\sqrt{2}$ or $\sqrt{3}$. However, mathematics strives for generality because general methods and constructions are more powerful. This brought into existence the system of real numbers. Later, mathematicians found that the new structure also solves the problem of the topological completion of rational numbers, i.e., real numbers made rational numbers topologically complete.

Another deficiency with respect to algebraic equations was the impossibility to solve such simple equations as $x^2 + 1 = 0$ and similar ones. Elimination of this deficiency resulted in the creation of the system of all complex numbers. In such a way, one more "unsolvable" problem was solved.

Thus, we can see a recurring scheme across these situations. Namely, encountering problems that they were not able to solve and for which it had been proved that these problems were unsolvable, creative mathematicians introduced new structures, extending the existing ones and making it possible to solve previously "unsolvable" problems. For instance, it was proved that it was impossible to solve the equation $x^2 = 2$ with rational numbers and thus, with numbers that mathematicians knew at that time. However, creation of real numbers made possible to solve this and many other similar equations.

There are many examples of such situations. One of them is also related to numbers. When mathematicians started working with infinite systems, such as sets of all natural numbers or all real numbers, they needed efficient procedures and rules for operating with infinities. This also demanded new mathematical structures. At first, limits and methods of summation were introduced for infinite sets of numbers. Then a new effort of human intelligence was made and transfinite numbers were created by Georg Cantor. This resulted in set theory—the most popular foundation of mathematics (Kuratowski and Mostowski 1967). We may call this approach to infinity arithmetical as it extends the arithmetic of natural numbers to the arithmetic of transfinite numbers (Cantor 1883).

A similar problem of operating with infinities brought forth real and complex hypernumbers. Only in this case, infinities were coming not from abstract mathematics but from physics. Namely, an important class of problems that appears in contemporary physics and involves infinite values inspired the theory of hypernumbers and extrafunctions, which is the topic of this book. A lot of mathematical models that are used in modern field theories (e.g., gauge theories) imply divergence for analytically calculated properties of physical systems. A customary method of divergence elimination in renormalization is regularization, i.e., the elimination of infinity by introducing ad hoc regulators at problematic places in a given type of calculations (cf. Edzawa and Zuneto 1977; Collins 1984). Nevertheless, the majority of them were not well defined mathematically because they utilized operations that neither mathematicians nor physicists could rigorously derive from axioms that would reflect physical principles. Moreover, there are such models in physics that contain infinities that cannot be eliminated by existing mathematical and physical theories. Divergent processes are at the core of the classical and modern mathematical analysis. Careful control and management of these processes are essential (Bray and Stanojevic 1998).

As a result, the necessity of precise mathematical methods dealing with infinite values that come from physics emerges. The theory of hypernumbers and extrafunctions provides such possibilities. This theory emanated from physically directed thinking. In it, the universes $\boldsymbol{R}$ and $\boldsymbol{C}$ of real and complex numbers are expanded to the universes $\boldsymbol{R}_\omega$ and $\boldsymbol{C}_\omega$ of real and complex hypernumbers, correspondingly. As a result, divergent integrals and series that appear in calculations with physical quantities become correctly defined as exact mathematical objects. Moreover, in the universe of hypernumbers, all sequences and series of real and complex numbers, as well as definite integrals of arbitrary functions have definite values. For convergent sequences, series and integrals, these values are ordinary numbers, while for divergent sequences, series and integrals, these values are infinite and oscillating hypernumbers. As a consequence, control and management of divergent processes becomes much simpler. In addition, functional integrals are treated as ordinary integrals, in which hypermeasures are used instead of ordinary measures (Burgin 1990, 2008/2009). Besides, in the context of the theory of hypernumbers and extrafunctions, it is possible to differentiate any function and determine values of such derivatives at any point. How to do this is explained in this book. This power of theory enables one to find solutions to differential equations for which it can be proved that they do not have solutions even in the spaces of distributions (Burgin and Ralston 2004).

It is natural to compare real hypernumbers to other generalizations of real numbers. Considering number systems in which infinite numbers exist, it is necessary to mention hyperreal numbers from nonstandard analysis introduced by Robinson (1961, 1966), surreal numbers introduced by Conway (1976), superreal numbers introduced by Tall (1980), and generalized numbers introduced by Egorov (1990). The construction of surreal numbers (cf. Knuth 1974) is a synthesis of nonstandard hyperreal numbers and transfinite numbers.

A principal difference between hypernumbers and other mentioned number systems is that all of those systems contain infinitely small numbers, while there are no infinitely small hypernumbers. In the context of physics, absence of infinitely small numbers is a definite advantage because physicists never encounter infinitely small numbers in their studies, both in experimental and theoretical physics. For instance, in physical measurements, there are no such things as infinitely small values, instead there is either a finite measured magnitude or it is zero. At the same time, the problem of divergence, bringing infinity in the theoretical realm of physics, haunted quantum theory for a long time. For instance, Kauffman (2001) writes:

> [Feynman] integrals led to mathematical and physical problems. On the mathematical side the integrals do not always have an associated measure theory. A mathematician would tend to say that the integrals do not exist. On the physical side, once it was a given that one had to sum over all possible interactions to have the state of (say) an electron, then that electron became surrounded by a cloud of (virtual) photons, indicators of the way the electron interacted with its own electromagnetic field. The summations, as a rule, diverged to infinity. Various "renormalization schemes" came into being to tame these self-generated infinities of quantum electrodynamics. All quantum field theories beyond a certain complexity require such renormalization, and the problem of making mathematical sense of these matters of integration and renormalization continues to the present day.

An early example of a problematic infinity in physics is the self-energy of an electron in quantum electrodynamics. Being a point mass, the electron particle should in theory have an infinite amount of self-energy through vacuum polarization, and therefore be infinitely heavy. This is obviously not the case in nature, which led researchers to introduce the concept of *renormalization* that would by definition assign a finite mass and finite charge to the electron. With this definition and corresponding procedure quantum electrodynamics today describes the fundamental forces in nature with the highest precision, by wide margin. Yet renormalization is motivated only heuristically from direct measurement, but not from physical principles or mathematical axioms. In addition, the renormalization procedures only work for a certain class of quantum field theories, called renormalizable quantum field theories.

As a result, some of the top physicists were dissatisfied with this situation. As Jackiw (1999) writes:

> The apparently necessary presence of ultraviolet infinities has dismayed many who remain unimpressed by the pragmatism of renormalization: Dirac and Schwinger, who count among the creators of quantum field theory and renormalization theory, respectively, ultimately rejected their constructs because of the infinities.

For instance, Dirac (1963) wrote:

> It is because of the good agreement with experiments that physicists do attach some value to the renormalization theory, in spite of its illogical character.

To this, he added:

> It was always possible for the pure mathematician to come along and make the [physical] theory sound by bringing in further steps, and perhaps by introducing quite a lot of cumbersome notation and other things that are desirable from a mathematical point

> of view in order to get everything expressed rigorously but do not contribute to the physical ideas. The earlier mathematics could always be made sound in that way, but in the renormalization theory we have a theory that has defied all the attempts of the mathematician to make it sound.

Later Dirac expressed his dissatisfaction in the following words (cf. Kragh 1990):

> Most physicists are very satisfied with the situation [with renormalization]. They say: 'Quantum electrodynamics is a good theory and we do not have to worry about it any more.' I must say that I am very dissatisfied with the situation, because this so-called 'good theory' does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small—not neglecting it just because it is infinitely great and you do not want it!

The opinion of another Nobel Prize winner Julian Schwinger was similar (Schwinger 1982).

One more important critic was Richard Feynman. Despite his crucial role in the development of quantum electrodynamics, he wrote (Feynman 1985):

> The shell game that we play . . . is technically called 'renormalization'. But no matter how clever the word, it is still what I would call a dippy process! Having to resort to such hocus-pocus has prevented us from proving that the theory of quantum electrodynamics is mathematically self-consistent. It's surprising that the theory still hasn't been proved self-consistent one way or the other by now; I suspect that renormalization is not mathematically legitimate.

Having a similar opinion and referring to the question of mathematical consistency for the whole renormalization program, and the ability to (reliably) calculate nuclear processes in quantum chromodynamics, Dyson (1996) wrote: ". . . in spite of all the successes of the new physics, the two questions that defeated me in 1951 remain unsolved."

Many efforts have been made to eliminate infinity from quantum theory. The main argument for this was based on the experience telling that people have never encountered infinity in their practice. Some of these efforts, such as quantization or distribution theory, looked quite natural and advanced physical understanding. Others looked like artificial ad hoc tricks. As a result, physicists started thinking about discovering an accurate theory that does not have renormalization problems. For instance, Ryder (1996) writes:

> In the Quantum Theory, these [classical] divergences do not disappear; on the contrary, they appear to get worse. And despite the comparative success of renormalization theory the feeling remains that there ought to be a more satisfactory way of doing things.

There was a hope that string theory, which is an outgrowth of quantum field theory, would eliminate all divergencies (Deligne et al. 1999). Unfortunately, the strings under scrutiny are almost exclusively those defined in 10 space-time dimensions. Recently these strings were discovered to be membranes in disguise, defined in 11 dimensions. These theories have the problem that membranes are infinite even in perturbation theory. Infinity of membranes corresponds to infinities appearing only nonperturbatively in 10 dimensional strings, and thus previously unrecognized (Siegel 1988).

Regular reappearance of infinite values in physical theories from the beginning of the twentieth century and to our days allows one to make a conjecture that infinity is inherent to the theoretical worldview and it is necessary only to learn how to deal with it. Indeed, some researchers suggested that singularities are certainly essential for explanatory purposes in quantum physics, being often important sources of information about the world. For instance, Jackiw (1999) argues that the divergences of quantum field theory must not be viewed as unmitigated defects, but on the contrary, they convey crucially important information about the physical situation, without which most of our theories would not be physically acceptable. In a similar way, Batterman (2009) demonstrates that a potential finite theory, free of the infinities that plague quantum field theory and condensed matter physics, cannot explain the existence of emergent protected states of matter. However, to unfailingly preserve infinities in a physical theory, it is necessary to have mathematical tools to consistently work with infinities and divergences.

To formally represent some kinds of such singularities, physicists introduced such tools as Heaviside-Dirac delta function, which was used by them without relevant mathematical foundation. This situation was cured by distribution theory. It taught physicists how to rigorously work with some functions that take infinite values, such as delta-function, bringing new tools and insight to the physical theory. The theory of hypernumbers and extrafunctions continues this line of reasoning, extensively extending the scope of distribution theory and providing prospective means for improving exactness and capability of physical theories. Hypernumbers studied in this book allow physicists to use infinite quantities in models of physical phenomena. They can work in this situation describing the physical realm better than any before-mentioned number system, as they avoid introduction of unphysical spaces.

It is necessary to remark that real hypernumbers are sometimes confused with hyperreal numbers from nonstandard analysis, which was created to solve an important problem of the calculus (Robinson 1966). As we know, the calculus created by Leibnitz and Newton used infinitesimals or infinitely small quantities in mathematics. Leibnitz called them *differentials* and Newton called them *fluents* and *fluxions*. Infinitesimals appeared in seventeenth century as extremely useful but "illegal" mathematical entities. Leibnitz was severely criticized for his usage of infinitesimals, and Newton in his works even stressed that he gave up these "illegal" infinitesimals (Kline 1980). However, in spite of all criticism, methods based on infinitely small quantities gave valid results. Moreover they were very efficient. That is why, different mathematicians tried to elaborate sound foundations for such methods. Only the outstanding French mathematician Cauchy did it in the nineteenth century. He justified methods based on infinitely small quantities. However, infinitely small quantities were interpreted in another sense in comparison with the interpretation of Leibnitz. It was not possible to operate with them in the same manner as with natural numbers. They did not become exact mathematical constructions. Only in the second part of the twentieth century exact mathematical constructions were related to infinitely small quantities in the nonstandard analysis by Robinson (1961, 1966). We may call his approach set-theoretical because the main construction of nonstandard numbers is based on set-theoretical axioms. This explains why nonstandard analysis is not sufficiently relevant to physical problems. Nonstandard numbers change the spaces of real and complex numbers because between any two real (complex) numbers a multitude of infinitely small numbers is inserted. At the same time, infinitely small numbers never appear in physics. In contrast to this, hypernumbers are built essentially from a topological perspective preserving the topology of real and complex numbers.

Another generalization of real numbers (as well as of ordinal numbers) is surreal numbers (Knuth 1974; Conway 1976). They are extremely general and contain hyperreal numbers, superreal numbers (Tall 1980), and ordinal numbers (Cantor 1932) as subclasses. As Ehrlich (1994, 2001) writes, the system of surreal numbers contains "all numbers great and small." So, it is possible to ask a question whether hypernumbers are some kinds of surreal numbers. The answer is negative because surreal numbers cannot contain hypernumbers that are not separated from zero as is proved in Sect. 2.2.

Generalized numbers of Egorov (1990) are similar to the hyperreal numbers from nonstandard analysis, but their construction is simpler—instead of ultrafilters, the filter of all cofinite sets is utilized for the factorization of the space of all real sequences. However, this simplicity results in a worse topology in the space of generalized numbers in comparison with the topology in the space of hypernumbers.

It is necessary to understand that when we compare hypernumbers with hyperreal numbers from nonstandard analysis or with transfinite numbers from set theory, we do not try to assert that hypernumbers are better. Our goal is to show that all these classes of numbers are necessary but each of them is more adequate for a specific area. Transfinite numbers were introduced for and work well in foundations of mathematics (Cantor 1883, 1932; Kuratowski and Mostowski 1967). Hyperreal numbers were invented for and work well in foundations of the calculus (Robinson 1966). Real and complex hypernumbers are oriented to problems of physics.

1.1 The Structure of this Book

In Chap. 2, real hypernumbers are introduced and their properties studied. In a similar way, it is possible to build complex hypernumbers and study their properties (Burgin 2002, 2004, 2010). In Sect. 2.1, the construction and some characteristics of real hypernumbers are described. In Sect. 2.2, algebraic properties of hypernumbers are investigated. In particular, it is proved that the set $\boldsymbol{R}_\omega$ of all real hypernumbers is an ordered linear space over the field $\boldsymbol{R}$ of real numbers. In Sect. 2.3, topological properties of hypernumbers are explored. In particular, it is proved that the space of all real hypernumbers is Hausdorff. This provides uniqueness for limits what is very important for analysis. In addition to this, it is proved that hypernumbers are determined by a topological invariance principle: the space of all real hypernumbers is the largest Hausdorff quotient of the sequential extension of the space of all real numbers. It is necessary to remark that in this book, we consider only the simplest case of hypernumbers. More general classes of hypernumbers are introduced and explored by Burgin (2001, 2005c).

In Chap. 3, we define and study various types of extrafunctions. The main emphasis is on general extrafunctions and norm-based extrafunctions, which include conventional distributions, hyperdistributions, restricted pointwise extrafunctions, and compactwise extrafunctions. In Sect. 3.1, the main constructions are described and their basic properties are explicated. In Sect. 3.2, various algebraic properties of norm-based extrafunctions are obtained. For instance, it is demonstrated that norm-based extrafunctions form a linear space over the field of real numbers $\boldsymbol{R}$. It is also proved that the class of all bounded norm-based extrafunctions is a linear algebra over $\boldsymbol{R}$.

An important property of mathematical spaces used for modeling physical systems is their topology. As the history of physics shows, the topology of underlying spaces is inherently connected with properties of physical systems (cf. Nash 1997; Witten 1988b). An important field of modern quantum physics is formed by topological quantum field theories (Atiyah 1988; Witten 1988a). An inappropriate topology in the state space can result in insolvability of such simple partial differential equations as $\partial/_{\partial t} f = c$ where c is a constant (Oberguggenberger 1992). That is why in Sect. 3.3, we study topological properties of norm-based extrafunctions, demonstrating that norm-based extrafunctions form a Hausdorff topological space.

In Chap. 4, basic elements of the *theory of hyperdifferentiation*, also called the *extended differential calculus*, are presented. It is demonstrated what advantages hypernumbers and extrafunctions possess for differentiation of real functions. In Sect. 4.1, basic elements of the theory of approximations are presented. In Sect. 4.2, we introduce and study properties of sequential partial derivatives of real functions. Some of these properties are similar to properties of conventional derivatives, while others are essentially different.

Here we consider only extended differentiation, called hyperdifferentiation, of real functions. Differentiation of extrafunctions is studied in Burgin (1993, 2002), while hyperdifferentiation and differentiation of complex functions is studied in Burgin and Ralston (2004) and Burgin (2010).

Chapter 5 presents basic elements of the *theory of hyperintegration*, also called the *extended integral calculus*, demonstrating how hypernumbers and extrafunctions increase the power of integration. Parallel to partial differentiation, here we introduce partial integration and hyperintegration. Partial integration is well defined for partially defined functions and should be distinguished from integration by parts. Using hypernumbers, we define *partial Riemann hyperintegrals and integrals*. One the one hand, partial Riemann hyperintegrals always exist for any total in an interval real function. Moreover, although Shenitzer and Steprāns (1994) write, "There is no perfect integral, one which would make all functions integrable," here we show that extending the concept of integral to the concept of partial hyperintegral, it becomes possible to integrate any totally defined real function. On the other hand, it is demonstrated that not only the conventional

Riemann integrals but also gauge integrals and Lesbegue integrals are special cases of partial Riemann hyperintegrals. This shows that the concept and structure of hyperintegration is a natural extension of the concept and structure of conventional integration. Moreover, many properties of integrals are preserved by this extension. For instance, it is proved that any partial Riemann hyperintegral is a linear hyperfunctional and any partial Riemann integral is a linear functional in the space of all real functions. At the same time, there are properties of partial hyperintegrals that are essentially different. For instance, as it is demonstrated in Sect. 5.2, there are real functions f such that for any interval $[a, b]$ and any real hypernumber α and in particular, for any real number a, there is a partial Riemann hyperintegral of f equal to α (to a). It is necessary to remark that in this book, we consider only hyperintegrals of real functions, which are linear hyperfunctionals. More general hyperfunctionals are studied by Burgin (1991, 2004), while hyperintegration in more general spaces (in particular, in infinite dimensional spaces) is studied by Burgin (1995, 2004, 2005a).

To make the book easier for the reader, notations for and definitions of the main mathematical concepts and structures used in the book are included in Appendix.

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Chapter 2
Hypernumbers

I realize that in this undertaking I place myself in a certain opposition to views widely held concerning the mathematical infinite and to opinions frequently defended on the nature of numbers.

(Georg Cantor 1845–1918)

In this chapter we introduce real hypernumbers and study their properties in Sect. 2.1. Algebraic properties are explored in Sect. 2.2, and topological properties are investigated in Sect. 2.3. In a similar way, it is possible to build complex hypernumbers and study their properties (Burgin 2002, 2004, 2010).

2.1 Definitions and Typology

Let $\boldsymbol{a} = (a_i)_{i\in\omega}$ be a sequence of rational numbers, which is a mapping $f_a : \omega \to \boldsymbol{Q}$ where ω is the totally ordered set $\{1, 2, 3, \ldots, n, \ldots\}$ and $\boldsymbol{Q}^\omega = \{(a_i)_{i\in\omega}; a_i \in \boldsymbol{Q}\}$ be the set of all sequences of rational numbers $\boldsymbol{Q}$.

Definition 2.1.1 For arbitrary sequences $\boldsymbol{a} = (a_i)_{i\in\omega}$ and $\boldsymbol{b} = (b_i)_{i\in\omega}$ from $\boldsymbol{Q}^\omega$,

$$\boldsymbol{a} \approx \boldsymbol{b} \textit{ means} \text{ that } \lim_{i\to\infty}|a_i - b_i| = 0$$

Lemma 2.1.1 *The relation* $\approx$ *is an equivalence relation in* $\boldsymbol{Q}^\omega$.

Indeed, by definition, this relation is reflexive and symmetric. Thus, we need only to show that it is transitive. Taking three sequences $\boldsymbol{a} = (a_i)_{i\in\omega}$, $\boldsymbol{b} = (b_i)_{i\in\omega}$, and $\boldsymbol{c} = (c_i)_{i\in\omega}$ from $\boldsymbol{Q}^\omega$ such that $\boldsymbol{a} \approx \boldsymbol{b}$ and $\boldsymbol{b} \approx \boldsymbol{c}$, we have

$$\lim_{i\to\infty}|a_i - b_i| = 0$$

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and

$$\lim_{i\to\infty}|b_i - c_i| = 0$$

By properties of absolute values and limits, we have

$$0 \le \lim_{i\to\infty}|a_i - c_i| = \lim_{i\to\infty}|a_i - b_i + b_i - c_i|$$
$$\le \lim_{i\to\infty}|a_i - b_i| + \lim_{i\to\infty}|b_i - c_i| = 0 + 0 = 0$$

Consequently,

$$\lim_{i\to\infty}|a_i - c_i| = 0$$

i.e., $\boldsymbol{a} \approx \boldsymbol{c}$.

Definition 2.1.2 Classes of the equivalence $\approx$ are called *real hypernumbers* and their set is denoted by $\boldsymbol{R}_\omega$.

Any sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ of rational numbers determines (and represents) a real hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ and such a sequence is called a *defining sequence* or *representing sequence* or *representation* of the hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$. Definition 2.1.2 means that any two hypernumbers $\mathrm{Hn}(a_i)_{i\in\omega}$ and $\mathrm{Hn}(b_i)_{i\in\omega}$ are equal if their defining sequences $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ are equivalent.

Lemma 2.1.2 *If* $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$, $\beta = \mathrm{Hn}(b_i)_{i\in\omega}$, *all elements* a_i *and* b_i *are integer numbers and* $a_i \neq b_i$ *for infinitely many of them, then* $\alpha \neq \beta$.

Proof is left as an exercise.

This construction of hypernumbers is similar (almost the same) to the construction of real numbers by taking equivalence classes of Cauchy sequences of rational numbers. The only but essential difference is that here we take all sequences of rational numbers. As the equivalence relation is the same in both cases, we have the following result. Let us consider the following example.

Example 2.1.1 For real numbers, we have $1 = 0.99999\ldots9$. The same is true for hypernumbers, i.e., $1 = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = 0.999\ldots9$ where 9 is repeated i times, i.e., the sequence 0.9, 0.99, 0.999, … represents the real hypernumber 1, which is also a real number as we can see from the following statement.

Proposition 2.1.1 *The set* $\boldsymbol{R}$ *of all real numbers is a subset of the set* $\boldsymbol{R}_\omega$ *of all real hypernumbers*.

Thus, by the described construction, we obtain both all real numbers and many real hypernumbers that are different from real numbers, for example, infinite hypernumbers, such as $\alpha = \mathrm{Hn}(i)_{i\in\omega}$. At the same time, it is possible to build real hypernumbers using sequences of real numbers.

Let $\boldsymbol{R}^{\omega} = \{(a_i)_{i\in\omega}; a_i \in \boldsymbol{R}\}$ be the set of all sequences of real numbers. We remind that a sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ of real numbers is a mapping $f_a : \omega \to \boldsymbol{R}$.

Definition 2.1.3 For arbitrary sequences $\boldsymbol{a} = (a_i)_{i\in\omega}$ and $\boldsymbol{b} = (b_i)_{i\in\omega}$ from $\boldsymbol{R}^{\omega}$,

$$\boldsymbol{a} \sim \boldsymbol{b} \text{ means that } \lim_{i\to\infty}|a_i - b_i| = 0$$

The introduced relation ~ is an equivalence relation (Burgin 2002).

The set of all classes of the equivalence ~ is denoted by $\mathbf{R}_{\omega}$. Elements of this set are also called real hypernumbers Burgin (1990, 2002, 2004, 2005a). There are definite reasons to do this as we explain below (cf. Theorem 2.1.1).

Lemma 2.1.3 *The set* $\boldsymbol{R}$ *is isomorphic to the subset* $\underline{\boldsymbol{R}} = \{\alpha = \mathrm{Hn}(a_i)_{i\in\omega}; a_i = a \in \boldsymbol{R}$ for all $i \in \omega\}$ of the set $\mathbf{R}_{\omega}$.

Proof is left as an exercise.

When it does not cause confusion, we call elements from $\underline{\boldsymbol{R}}$ real numbers and denote this set by $\boldsymbol{R}$. In what follows, we will identify hypernumbers from $\underline{\boldsymbol{R}}$ and corresponding real numbers, assuming that $\boldsymbol{R} = \underline{\boldsymbol{R}}$ is a subset of $\boldsymbol{R}_{\omega}$. For instance, Hn(8, 8, 8, ..., 8, ...) = 8 in $\boldsymbol{R}_{\omega}$.

As we know not all real numbers are also rational numbers and consequently, not all sequences of real numbers are also sequences of rational numbers. There are much more real numbers, which form a continuum, than rational numbers, which form a countable set. However, there is a one-to-one correspondence between all introduced equivalence classes of real sequences and all introduced equivalence classes of rational sequences. It means that we have the following result.

Theorem 2.1.1 *There is a one-to-one correspondence between sets* $\mathbf{R}_{\omega}$ *and* $\boldsymbol{R}_{\omega}$, *i.e., these sets are equipotent.*

Proof At first, we show that any class of equivalent real sequences contains a class of equivalent rational sequences. Indeed, let us take a class α of equivalent real sequences and a real sequence $(a_i)_{i\in\omega}$ that belongs to this class. As the set $\boldsymbol{Q}$ of all rational numbers is dense in the set $\boldsymbol{R}$ of all real numbers, for each number $a_i (i \in \omega)$, there is a rational number b_i, such that

$$|a_i - b_i| < 1/i$$

By Definition 2.1.3, the rational sequence $(b_i)_{i\in\omega}$ belongs to the class α and thus, the whole class of rational sequences equivalent to the sequence $(b_i)_{i\in\omega}$ is a subset of the class α.

This gives us a mapping f of the set $\boldsymbol{R}_{\omega}$ onto the set $\mathbf{R}_{\omega}$. Besides, two different classes from the set $\boldsymbol{R}_{\omega}$ cannot be subsets of the same class in the set $\mathbf{R}_{\omega}$ because the relation $(c_i)_{i\in\omega} \sim (d_i)_{i\in\omega}$ implies the relation $(c_i)_{i\in\omega} \approx (d_i)_{i\in\omega}$ for any rational sequences $(c_i)_{i\in\omega}$ and $(d_i)_{i\in\omega}$.

Consequently, f is a one-to-one mapping, which means equipotence of the sets $\mathbf{R}_{\omega}$ and $\boldsymbol{R}_{\omega}$.

Theorem is proved.

Even more, as it is demonstrated in Burgin (2004), the spaces $\mathbf{R}_\omega$ and $\boldsymbol{R}_\omega$ are isomorphic as topological spaces and as linear spaces.

So, it is natural to assume that both sets $\boldsymbol{R}_\omega$ and $\mathbf{R}_\omega$ represent the same object, which is called the set of all real hypernumbers. It means that rational numbers and real numbers as building blocks for real hypernumbers give us the same construction. Consequently, any sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ of real numbers determines (and represents) a real hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ and such a sequence is called a *defining sequence* or *representing sequence* or *representation* of the hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$.

However, each of the approaches to the hypernumber construction has its advantages. When we build hypernumbers from rational numbers, we use finite constructive elements, while this is not true for the second construction. Indeed, in contrast to rational numbers, real numbers are inherently infinite because they emerge as a result of an infinite process or as a relation in infinite sets. In addition, when the first construction is used, it is not necessary to construct separately real numbers—they automatically emerge as a subclass of real hypernumbers. At the same time, when we construct real hypernumbers from real numbers, it makes the construction more transparent and helps to better understand properties and behavior of hypernumbers. That is why here we assume that hypernumbers are generated by real numbers and any sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ of real numbers determines/represents a hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$.

In the same way, it is possible to build *complex hypernumbers*. They are equivalence classes of sequences of complex numbers and form the set $\boldsymbol{C}_\omega$ of complex hypernumbers.

Some may be appalled by transparent infinity of many hypernumbers. However, infinity is already inherent to the majority of real numbers and these numbers are very useful to people. Indeed, an explicit digital representation of such numbers as π or e demands infinite number of digits. This situation worried mathematicians who were suspicious of infinity as an actual object. For instance, Borel (1952) devoted a whole book to discuss these problems of the existence of real numbers, in particular, the problem of "inaccessible numbers." As Borel defines, an accessible number is a number that can be described as a mathematical object. The problem is that it is permissible only to use a finite process or finite description of a process to describe a real number. Thus, it is possible to describe integer numbers easily enough by an algorithm that generates the decimal representation of an arbitrary integer number. An algorithmic description of integer numbers allows us to do the same with rational numbers. For instance, it is possible to represent a rational number either as a pair of integer numbers or by specifying the repeating decimal expansion. Hence, integer and rational numbers are accessible. We can do the same with some real numbers but not with all. For example, if we take the, so called, Liouville transcendental number, then it can be described (built) by an algorithm that puts 1 in the place $n!$ and 0 elsewhere. A finite way of specifying the nth term in a Cauchy sequence of rational numbers gives us a finite description of the resulting real number. However, as Borel pointed out, there are a countable number of such descriptions or algorithms. At the same time, there are uncountably many (the continuum) real numbers and the continuum is much bigger than any countable number. So, the natural conclusion is that the majority of real numbers are inaccessible and thus, as Borel claimed, it is impossible to operate with them.

Besides, Borel (1927) pointed out that when we consider a real number as an infinite sequence of digits, then we could put an infinite amount of information into a single number, building the "know-it-all" number. First, it is possible to build a real number q such that q contains every English sentence in a coded form. Each sentence is coded by a block of decimal digits. All English sentences are ordered and these blocks go after the decimal point in q one after another in the same order. In particular, q contains every possible true/false question that can be asked in English. Then we construct a real number r as follows. If the nth block of q translates into a true/false question, then we set the nth digit of q after the decimal point equal to 1 if the answer to the question is true and equal to 2 if the answer is false. If the nth block of q does not translate into a true/false question, then we set the nth digit of r after the decimal point is equal to 3. Thus, using r, it is possible to answer every possible question that has ever been asked, or ever will be asked, in English. Borel calls this number r an unnatural real number, or an "unreal" real. This argument was further developed in Burgin (2005b). It was demonstrated that a possibility to operate with arbitrary real numbers makes it possible to compute any function defined for words in a finite alphabet.

Thus, there are problems with inherent infinity in transcendental real numbers, but in spite of this these numbers are efficiently used in many areas. In the same way, hypernumbers could be useful in many areas and specifically in physics.

Definition 2.1.4 A real hypernumber α is *represented* in a set $X \subseteq \boldsymbol{R}$ if there is a sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ of real numbers such that $a_i \in X$ for all $i \in \omega$ and $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$.

For instance, all real hypernumbers are represented in the set $\boldsymbol{Q}$ of all rational numbers.

Real numbers contain different subclasses, such as rational numbers, irrational numbers, transcendental numbers, integer numbers, etc. In the universe of hypernumbers $\boldsymbol{R}_\omega$, there are even more distinct subclasses. As $\boldsymbol{R} \subseteq \boldsymbol{R}_\omega$, the set $\boldsymbol{R}_\omega$ also contains all subclasses of $\boldsymbol{R}$. However, there are also many new subclasses. For instance, it is possible to introduce three types of hypernumbers: stable, infinite, and oscillating hypernumbers.

Example 2.1.2 An infinite increasing hypernumber: $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = i$, $i = 1, 2, 3, \ldots$.

Example 2.1.3 An infinite increasing hypernumber: $\beta = \mathrm{Hn}(b_i)_{i\in\omega}$ where $b_i = 2^i$, $i = 1, 2, 3, \ldots$.

Example 2.1.4 A finite (bounded) oscillating hypernumber: $\gamma = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = (-1)^i$, $i = 1, 2, 3, \ldots$.

Example 2.1.5 An infinite oscillating hypernumber: $\delta = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = (-1)^i \cdot i$, $i = 1, 2, 3, \ldots$.

Example 2.1.6 An infinite decreasing hypernumber: $\nu = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = -5i$, $i = 1, 2, 3, \ldots$

Example 2.1.7 An infinite oscillating hypernumber: $\theta = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = 2^i + (-1)^i \cdot i$, $i = 1, 2, 3, \ldots$.

Let us give exact definitions for these classes of hypernumbers.

Definition 2.1.5 A sequence $(a_i)_{i\in\omega}$ is called *bounded* if there is a real number b and a natural number n such that $|a_i|<b$ for all $i > n$.

Bounded sequences from $\boldsymbol{R}^\omega$ are mapped onto finite hypernumbers from $\boldsymbol{R}_\omega$.

Definition 2.1.6 A real hypernumber α is called *finite* or *bounded* if there is a sequence $(a_i)_{i\in\omega}$ such that $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ and for some positive real number a, we have $|a_i| < a$ for almost all $i \in \omega$. The set of all finite/bounded real hypernumbers is denoted by $\mathbf{F}\boldsymbol{R}_\omega$.

Proposition 2.1.2 *The following conditions are equivalent:*

(a) α is a finite real hypernumber
(b) There is a sequence $(a_i)_{i\in\omega}$ such that $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ and for some real number a, $|a_i|<a$ *for all* $i \in \omega$
(c) For any sequence $(a_i)_{i\in\omega}$ such that $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$, there is a real number a such that $|a_i|<a$ for almost all $i \in \omega$
(d) α is represented in some interval

Proof (a) $\Leftrightarrow$ (b). If $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ and $|a_i| < a$ for almost all $i \in \omega$. It means that in the sequence $(a_i)_{i\in\omega}$, there is only a finite number of elements a_i such that the absolute value of each of them is larger than a. Then taking b equal to $c + 1$ where c is the maximum the absolute values of all elements from the sequence $(a_i)_{i\in\omega}$, we have $|a_i| < b$ for all $i \in \omega$. Thus, (a) implies (b) and naturally, (b) implies (a) as (b) is a stronger condition.

(a) $\Leftrightarrow$ (c). Let us assume that $\alpha = \mathrm{Hn}(a_i)_{i\in\omega} = \mathrm{Hn}(b_i)_{i\in\omega}$ and for some positive real number a, we have $|a_i| < a$ for almost all $i \in \omega$. By Definition 2.1.2, the sequences $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ are equivalent. It means that $\lim_{i\to\infty}|a_i - b_i| = 0$. Thus, if the absolute values of almost all a_i are bounded, then the absolute values of almost all b_i are also bounded. Consequently, (a) implies (c) and naturally, (c) implies (a) as (c) is a stronger condition.

Equivalence of conditions (b) and (c) is also true as any equivalence relation is transitive, e.g., (b) $\Leftrightarrow$ (a) and (a) $\Leftrightarrow$ (c) imply (b) $\Leftrightarrow$ (c).

In addition, conditions (b) and (d) are also equivalent because if $|a_i| < a$ for all $i \in \omega$, then α is represented in the interval $[-a, a]$, and if α is represented in the interval $[d, a]$, then $|a_i| < \max\{|a|, |d|\} + 1$ for all $i \in \omega$.

Proposition is proved.

Definition 2.1.7 A real hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ is called *proper* if $\alpha \notin \boldsymbol{R}$.

Improper real hypernumbers are exactly real numbers.

Definition 2.1.8 A real hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ is called *stable* if there is a real number b such that $\alpha = \mathrm{Hn}(b_i)_{i\in\omega}$ and $b_i = b \in \boldsymbol{R}$ for almost all $i \in \omega$.

For such a hypernumber, we have $\alpha = b \in \boldsymbol{R}$.

Definitions 2.1.6 and 2.1.8 imply the following result.

Lemma 2.1.4 *Any stable hypernumber* $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ *is finite.*

Proposition 2.1.3 *The following conditions are equivalent:*

(a) α is a stable real hypernumber
(b) There is a sequence $(b_i)_{i\in\omega}$ such that $\alpha = \mathrm{Hn}(b_i)_{i\in\omega}$ and for some real number b, we have $\mathrm{b_i} = \mathrm{b}$ for all $i \in \omega$
(c) There is a real number a such that for any sequence $(c_i)_{i\in\omega}$ such that $\alpha = \mathrm{Hn}(c_i)_{i\in\omega}$, we have $\lim_{i\to\infty} c_i = a$
(d) $\alpha \in \boldsymbol{R}$
(e) For some sequence $(c_i)_{i\in\omega}$ such that $\alpha = \mathrm{Hn}(c_i)_{i\in\omega}$, the limit $\lim_{i\to\infty} c_i$ exists and is equal to α

Proof If $\alpha = \mathrm{Hn}(b_i)_{i\in\omega}$ and $b_i = b$ for almost all $i \in \omega$. Then by Definition 2.1.2, $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = b$ for all $i \in \omega$. Thus, condition (a) implies condition (b) and naturally, condition (b) implies condition (a) as (b) is a stronger condition.

If α is a stable real hypernumber, then by Definition 2.1.8, there is a real number a and a sequence $(b_i)_{i\in\omega}$ such that $\alpha = \mathrm{Hn}(b_i)_{i\in\omega}$ and $b_i = a$ for almost all $i \in \omega$. At the same time, as $\mathrm{Hn}(b_i)_{i\in\omega} = \mathrm{Hn}(c_i)_{i\in\omega}$, we have by Definitions 2.1.1 and 2.1.2, that $\lim_{i\to\infty}|c_i - b_i| = 0$. Thus, $\lim_{i\to\infty} c_i = a$, and condition (a) implies condition (c).

If $\lim_{i\to\infty} c_i = a$ and in a sequence $(b_i)_{i\in\omega}$ almost all $b_i = b$, then $\mathrm{Hn}(b_i)_{i\in\omega} = \mathrm{Hn}(c_i)_{i\in\omega}$ is a stable hypernumber. Thus, condition (c) implies condition (a).

If $\alpha = \mathrm{Hn}(b_i)_{i\in\omega} = \mathrm{Hn}(c_i)_{i\in\omega}$ and the limit $\lim_{i\to\infty} a_i$ exists, then by Definitions 2.1.1 and 2.1.2, $\lim_{i\to\infty}|c_i - a_i| = 0$. Thus, $\lim_{i\to\infty} c_i = \lim_{i\to\infty} a_i$, and condition (e) implies condition (c). At the same time, condition (c) implies condition (e) as (c) is a stronger condition.

In addition, conditions (a) and (d) are equivalent by definition.

Proposition is proved.

Thus, it is possible to assume that stable real hypernumbers are real numbers, which belong to $\boldsymbol{R}_\omega$. In what follows, we will identify stable real hypernumbers and corresponding real numbers. For instance, Hn(10, 5, 1, 1, 1, 8, 8, …, 8, 8, 8, …) = 8 in $\boldsymbol{R}_\omega$.

Definition 2.1.9 A real hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ is called *oscillating* if there is a positive real number k such that there are two infinite sequences of natural numbers $m(i)$ and $n(i)$ with $i = 1, 2, 3, \ldots$ such that $m(i) < n(i) < m(i+1)$, $a_{m(i)} - a_{n(i)} > k$ and $a_{m(i+1)} - a_{n(i)} > k$ for all $i = 1, 2, 3, \ldots$

Remark 2.1.1 Oscillating hypernumbers may be bounded/finite (cf. Example 2.1.4) and unbounded or infinite (cf. Examples 2.1.5 and 2.1.7). In Examples 2.1.4, 2.1.5, and 2.1.7, $m(i) = 2i$ and $m(i) = 2i + 1$ $(i = 1, 2, 3, \ldots)$.

Note that oscillation in a hypernumber can be very fast as in the hypernumbers from Examples 2.1.4 and 2.1.5 or it can be very slow as in the hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = 10^{2i}$ when $10^{2i} \le i < 10^{2i+1}$ and $a_i = -10^{2i+1}$ when $10^{2i+1} \le i < 10^{2(i+1)}$.

Definition 2.1.10 A real hypernumber α is called:

1. *Increasing* if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ for some sequence $(a_i)_{i\in\omega}$ such that
$$\exists j \in \omega \quad \forall i > j(a_{i+1} - a_i \ge 0)$$
2. *Decreasing* if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ for some sequence $(a_i)_{i\in\omega}$ such that
$$\exists j \in \omega \quad \forall i > j(a_{i+1} - a_i \le 0)$$
3. *Strictly increasing* if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ for some sequence $(a_i)_{i\in\omega}$ such that
$$\exists j \in \omega \quad \forall i > j(a_{i+1} - a_i > 0)$$
4. *Strictly decreasing* if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ for some sequence $(a_i)_{i\in\omega}$ such that
$$\exists j \in \omega \quad \forall i > j(a_{i+1} - a_i < 0)$$
5. *Infinite increasing* if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ for some sequence $(a_i)_{i\in\omega}$ such that
$$\exists j \in \omega \quad \forall i > j(a_{i+1} - a_i \ge 0) \quad \text{and} \quad (\forall p \in \boldsymbol{R}^+ \exists\, i \in \omega(a_i > p))$$
6. *Infinite decreasing* if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ for some sequence $(a_i)_{i\in\omega}$ such that
$$\exists j \in \omega \quad \forall i > j(a_{i+1} - a_i \le 0) \quad \text{and} \quad (\forall p \in \boldsymbol{R}^+ \exists\, i \in \omega(a_i < -p))$$
7. *Infinite strictly increasing* if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ for some sequence $(a_i)_{i\in\omega}$ such that
$$\exists j \in \omega \quad \forall i > j(a_{i+1} - a_i > 0) \quad \text{and} \quad (\forall p \in \boldsymbol{R}^+ \exists\, i \in \omega(a_i > p))$$
8. *Infinite strictly decreasing* if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ for some sequence $(a_i)_{i\in\omega}$ such that
$$\exists j \in \omega \quad \forall i > j(a_{i+1} - a_i < 0)) \quad \text{and} \quad (\forall p \in \boldsymbol{R}^+ \exists\, i \in \omega(a_i < -p))$$
9. *Infinite expanding* if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ for some sequence $(a_i)_{i\in\omega}$ such that there are subsequences $(b_i)_{i\in\omega}$ and $(c_i)_{i\in\omega}$ of the sequence $(a_i)_{i\in\omega}$ such that $\mathrm{Hn}(b_i)_{i\in\omega}$ is an infinite increasing hypernumber and $\mathrm{Hn}(c_i)_{i\in\omega}$ is an infinite decreasing hypernumber

10. *Faithfully infinite expanding* if it is infinite expanding and the sequence $(a_i)_{i\in\omega}$ is the union of the sequences $(b_i)_{i\in\omega}$ and $(c_i)_{i\in\omega}$ such that $\mathrm{Hn}(b_i)_{i\in\omega}$ is an infinite increasing hypernumber and $\mathrm{Hn}(c_i)_{i\in\omega}$ is an infinite decreasing hypernumber
11. *Infinite monotone* if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ for some monotone sequence $(a_i)_{i\in\omega}$, i.e., $a_{i+1} \geq a_i$ for almost all i or $a_{i+1} \leq a_i$ for almost all i, that tends to infinity

It is possible to show that definitions of hypernumber classes are invariant with respect to the choice of sequences representing hypernumbers, i.e., a hypernumber stays in the same class when the representing sequence is changed.

Definition 2.1.10 directly implies the following result.

Lemma 2.1.5 *Infinite monotone hypernumbers are exactly infinite increasing hypernumbers or infinite decreasing hypernumbers.*

Infinite monotone hypernumbers form two classes:

– Positive infinite monotone hypernumbers, in which almost all elements of each representation are positive.
– Negative infinite monotone hypernumbers, in which almost all elements of each representation are negative.

Lemma 2.1.6
(a) Positive infinite monotone hypernumbers are exactly infinite increasing hypernumbers.
(b) Negative infinite monotone hypernumbers are exactly infinite decreasing hypernumbers.

Indeed, a representative of positive hypernumber cannot have elements that tend to negative infinity, while a representative of negative hypernumber cannot have elements that tend to positive infinity.

Lemma 2.1.7 *Any infinite expanding hypernumber is an unbounded oscillating hypernumber and vice versa.*

Proof
1. Let us take an infinite expanding hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$. Then there are subsequences $(b_i)_{i\in\omega}$ and $(c_i)_{i\in\omega}$ of the sequence $(a_i)_{i\in\omega}$ such that $\mathrm{Hn}(b_i)_{i\in\omega}$ is an infinite increasing hypernumber and $\mathrm{Hn}(c_i)_{i\in\omega}$ is an infinite decreasing hypernumber. Because the distance between elements b_i and c_i grows, there is a number $k \in \boldsymbol{R}^{++}$ such that $b_i - c_i > k$ and $b_{i+1} - c_i > k$ for all $i = 1, 2, 3, \ldots$. Indeed, as the sequence $(b_i)_{i\in\omega}$ is infinitely increasing, while the sequence $(c_i)_{i\in\omega}$ is decreasing, there is an element b_i larger than any element from the sequence $(c_i)_{i\in\omega}$ plus k. If $b_i = a_j$, we take $m(1) = j$.

   As the sequence $(c_i)_{i\in\omega}$ is infinite, we can find an element c_k such that $c_k = a_t$ and $t > j$. Then we take $n(1) = t$. As the sequence $(b_i)_{i\in\omega}$ is infinite, we can find an element b_l such that $b_l = a_q$ and $q > t$. Then we take $m(2) = q$. By construction, we have $a_{m(1)} - a_{n(1)} > k$ and $a_{m(2)} - a_{n(1)} > a_{m(1)} - a_{n(1)} > k$.

As the sequence $(c_i)_{i\in\omega}$ is infinite, we can find an element c_h such that $c_h = a_r$ and $r > q$. Then we take $n(2) = r$ and continue this process. It gives us two infinite sequences of natural numbers $m(i)$ and $n(i)$ with $i = 1, 2, 3, \ldots$ such that $m(i) < n(i) < m(i+1)$, $a_{m(i)} - a_{n(i)} > k$ and $a_{m(i+1)} - a_{n(i)} > k$ for all $i = 1, 2, 3, \ldots$. Consequently, by Definition 2.1.9, α is an unbounded oscillating hypernumber.

2. Let us take an unbounded oscillating hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$. As α is not bounded from above, the sequence $(a_i)_{i\in\omega}$ has a subsequence $(b_i)_{i\in\omega}$ that infinitely increases. As α is not bounded from below, the sequence $(a_i)_{i\in\omega}$ has a subsequence $(c_i)_{i\in\omega}$ that infinitely decreases. Consequently, α is an infinite expanding hypernumber.

   Lemma is proved.

The following lemma gives characteristic properties of real hypernumbers that are outside the real line.

Lemma 2.1.8 *A real hypernumber* $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ *is proper, that is,* $\alpha \notin \boldsymbol{R}$*, if and only if either the sequence* $\boldsymbol{a} = (a_i)_{i\in\omega}$ *is unbounded or (condition D) there is an interval* $(d, b) \subset \boldsymbol{R}$ *such that there are infinitely many elements* a_i *larger than* b *and infinitely many elements* a_i *smaller than* d.

Proof

Sufficiency. Let $\alpha \in \boldsymbol{R}$. Then by Lemma 2.1.3, $\lim_{i\to\infty} a_i = a \in \boldsymbol{R}$. Consequently (cf. Ross 1996), the sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ is bounded. Thus, if the sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ is unbounded, $a \notin \boldsymbol{R}$.

Let us assume that there is an interval $(d, b) \subseteq \boldsymbol{R}$ such that there are infinitely many elements a_i larger than b and infinitely many elements a_i smaller than d. Then taking some element a_i larger than b, we can find an elements a_j smaller than d with $j > i$. Then we can find an element a_t larger than b with $t > j$. Then we can find an element a_p smaller than d with $p > t$. Taking $k = b - d$, we see that this process will give us two infinite sequences of natural numbers $m(i)$ and $n(i)$ with $i = 1, 2, 3, \ldots$ such that $a_{m(i)} - a_{n(i)} > k$ and $a_{n(i)} - a_{m(i+1)} > k$ for all $i = 1, 2, 3, \ldots$. By Definition 2.1.9, α is an oscillating hypernumber. It means that condition D implies that α is a proper hypernumber.

Necessity. For any real hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$, we have two cases: the sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ is bounded or unbounded. If the sequence $\boldsymbol{a}$ is unbounded, then the first condition of the lemma is satisfied. So, to prove the lemma, we need to consider only the case when this sequence is bounded. It means that all elements a_i belong to some interval $[u, v]$.

Let us divide $[u, v]$ into three equal parts: $[u, u_1]$, $[u_1, u_2]$, and $[u_2, v]$. Then either (case 1) only one of these intervals (say, $[u, u_1]$) contains infinite number of elements a_i or (case 2) only two adjacent intervals (say, $[u, u_1]$ and $[u_1, u_2]$) contain infinite number of elements a_i or (case 3) both non-adjacent intervals $[u, u_1]$ and $[u_2, v]$ contain infinite number of elements a_i. In the latter case, everything is proved because it shows that α is an oscillating hypernumber with $k = b - d$ as we can take (u_1, u_2) as (d, b), validating condition D. Otherwise we continue decomposition of intervals: in the first case, of the interval $[u, u_1]$ and in the second case, of the interval $[u, u_2]$. Note that the length of the new interval obtained in the decomposition is equal, at most, to two thirds of the length of the interval that has been decomposed. We continue this process of decomposition.

If at some step of this process, we get the third case considered above, then we have an interval (d, b) that is necessary for validity of the lemma because it indicates that α is an oscillating hypernumber. If we always get cases one or two, then it gives us a system of nested closed intervals. The lengths of these intervals converge to 0 because the length of each of them is less than or equal to two thirds of the length of the previous interval and each of them contains almost all elements a_i. By the standard argument, this implies that the sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ has the limit. Consequently, $\alpha \in \boldsymbol{R}$.

Thus, if $a \notin \boldsymbol{R}$, then the condition of the lemma must be satisfied.

Lemma is proved.

Proposition 2.1.4 *Any finite real hypernumber is either a real number or an oscillating real hypernumber.*

Proof If a hypernumber α is defined by a bounded sequence $\boldsymbol{l}$, then $\boldsymbol{l}$ either converges or has, at least, two subsequences that converge to two different points. In the first case, α is a real number. In the second case, α is an oscillating real hypernumber because if a subsequence $\boldsymbol{h}$ of $\boldsymbol{l}$ converges to the point a, while a subsequence $\boldsymbol{g}$ of $\boldsymbol{l}$ converges to the point b and $a < b$, then the number $\frac{1}{2}(b - a)$ satisfies the condition for the number k from Definition 2.1.9.

Proposition is proved.

Corollary 2.1.1 *Any finite increasing (decreasing) real hypernumber is a real number.*

Definitions 2.1.9 and 2.1.10 imply the following result.

Proposition 2.1.5 *Any infinite real hypernumber is either an infinite increasing hypernumber or an infinite decreasing hypernumber or an oscillating real hypernumber.*

Proof If a hypernumber α is defined by a sequence $\boldsymbol{l}$, then being unbounded, $\boldsymbol{l}$ is either unbounded only above (case 1) or unbounded only below (case 2) or unbounded below and above (case 3). In the first case, there are two options for the hypernumber α: the sequence $\boldsymbol{l}$ either converges to ∞ or has two subsequences one of which converges to ∞, while the other converges to some real number a. In the first case, α is an infinite increasing hypernumber. The second option is true when in $\boldsymbol{l}$, there are infinitely many elements less than some real number c and it shows that α is an unbounded above oscillating real hypernumber because in this case, any positive real number satisfies the condition for the number k from Definition 2.1.9.

In the second case, which is symmetric to the first case, there are two options for α: the sequence $\boldsymbol{l}$ either converges to $-\infty$ or has a subsequence that either converges to $-\infty$, while another subsequence converges to some real number a. In the first case, α is an infinite decreasing hypernumber. The second option is true when in $\boldsymbol{l}$, there are infinitely many elements larger than some real number c and it shows that α is an unbounded below oscillating real hypernumber because any positive real number satisfies the condition for the number k from Definition 2.1.9.

In the third case, α is an oscillating infinite expanding real hypernumber because in this case, any positive real number satisfies the condition for the number k from Definition 2.1.9.

Proposition is proved.

Lemma 2.1.8 and Propositions 2.1.1 and 2.1.2 are used to give a complete characterization of real hypernumbers.

Theorem 2.1.2 *There are four disjoint classes of real hypernumbers and each hypernumber belongs to one of them:*

- Stable hypernumbers.
- Bounded/finite oscillating hypernumbers.
- Unbounded oscillating hypernumbers.
- Infinite monotone hypernumbers, which form two groups: increasing and decreasing hypernumbers.

In turn, unbounded oscillating hypernumbers form three groups:

- Bounded above oscillating hypernumbers.
- Bounded below oscillating hypernumbers.
- Unbounded below and above (infinite expanding) oscillating hypernumbers.

Proof Definitions 2.1.2 and 2.1.3 imply that any hypernumber α is either finite (bounded) or infinite. Indeed, if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega} = \mathrm{Hn}(b_i)_{i\in\omega}$, then either both sequences are bounded or both sequences are unbounded. In the first case, by Proposition 2.1.1, α is either a real number or an oscillating real hypernumber. In the second case, by Proposition 2.1.2, α is either an infinite increasing hypernumber or an infinite decreasing hypernumber or an unbounded oscillating real hypernumber.

In addition, Propositions 2.1.1 and 2.1.2 show that an oscillating hypernumber is either bounded or unbounded above or unbounded below or unbounded below and above (infinite expanding) oscillating hypernumber.

Theorem is proved.

Hypernumbers have specific invariants. One of the most important of them is the spectrum of a hypernumber. At first, we define the spectrum of a sequence.

Definition 2.1.11

(a) The *spectrum Spec* $\boldsymbol{a}$ of a sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ is the set

$$\{r \in \boldsymbol{R}; r = \lim_{i\to\infty} b_i \text{ for some subsequence } (b_i)_{i\in\omega} \text{ of } \boldsymbol{a}\}$$

(b) The *extended spectrum* E*Spec* $\boldsymbol{a}$ of a sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ is the set:

- *Spec* $\boldsymbol{a} \bigcup \{\infty, -\infty\}$ when $\lim_{i\to\infty} b_i = \infty$ for some subsequence $(b_i)_{i\in\omega}$ of $\boldsymbol{a}$ and $\lim_{i\to\infty} c_i = -\infty$ for some subsequence $(c_i)_{i\in\omega}$ of $\boldsymbol{a}$
- *Spec* $\boldsymbol{a} \bigcup \{\infty\}$ when $\lim_{i\to\infty} b_i = \infty$ for some subsequence $(b_i)_{i\in\omega}$ of $\boldsymbol{a}$ and there is no subsequence $(c_i)_{i\in\omega}$ of $\boldsymbol{a}$, for which $\lim_{i\to\infty} c_i = -\infty$
- *Spec* $\boldsymbol{a} \bigcup \{-\infty\}$ when $\lim_{i\to\infty} b_i = -\infty$ for some subsequence $(b_i)_{i\in\omega}$ of $\boldsymbol{a}$ and there is no subsequence $(c_i)_{i\in\omega}$ of $\boldsymbol{a}$, for which $\lim_{i\to\infty} c_i = \infty$
- *Spec* $\boldsymbol{a}$ in all other cases

Example 2.1.8 If $\boldsymbol{a} = (a_i)_{i \in \omega}$ where $a_i = \sin(i/2)\pi$, $i = 1, 2, 3, \ldots$, then $Spec\ a = \mathrm{E}Spec\ a = \{1, 0, -1\}$.

Example 2.1.9 If $\boldsymbol{a} = (a_i)_{i \in \omega}$ where $a_i = i$, $i = 1, 2, 3, \ldots$, then $Spec\ \boldsymbol{a} = \varnothing$, while $\mathrm{E}Spec\ \boldsymbol{a} = \{\infty\}$.

Note that if $\mathrm{E}Spec\ \boldsymbol{a} = \mathrm{E}Spec\ \boldsymbol{b}$, then $Spec\ \boldsymbol{a} = Spec\ \boldsymbol{b}$.

Lemma 2.1.9 *If* $\boldsymbol{a} = (a_i)_{i \in \omega} \sim \boldsymbol{b} = (b_i)_{i \in \omega}$, *then* Spec $\boldsymbol{a}$ = Spec $\boldsymbol{b}$ *and E*Spec $\boldsymbol{a}$ = *E*Spec $\boldsymbol{b}$.

Proof Let us consider a subsequence $\boldsymbol{d} = (d_i)_{i \in \omega}$ of the sequence $\boldsymbol{a} = (a_i)_{i \in \omega}$ defined by an injection $g : \omega \to \omega$, i.e., $d_i = a_{g(i)}$ for all $i = 1, 2, 3, \ldots$, and the corresponding subsequence $\boldsymbol{c} = (c_i)_{i \in \omega}$ of the sequence $\boldsymbol{b} = (b_i)_{i \in \omega}$ defined by the same mapping $g : \omega \to \omega$, i.e., $c_i = b_{g(i)}$ for all $i = 1, 2, 3, \ldots$. Then if

$$\lim_{i\to\infty} d_i = c \quad \text{for some real number } c,$$

the same is true for the sequence $\boldsymbol{c} = (c_i)_{i \in \omega}$, i.e.,

$$\lim_{i\to\infty} c_i = c$$

Indeed,

$$0 \leq \lim_{k\to\infty} \sup\{|a_{g(i)} - b_{g(i)}|; i > k\} \leq \lim_{k\to\infty} \sup\{|a_i - b_i|; i > k\} = 0$$

Thus,

$$\lim_{k\to\infty} \sup\{|a_{g(i)} - b_{g(i)}|; i > k\} = 0$$

and

$$\lim_{i\to\infty} |a_{g(i)} - b_{g(i)}| = 0$$

Consequently,

$$\lim_{i\to\infty} d_i = c$$

It means that if $c \in Spec\ \boldsymbol{a}$, then $c \in Spec\ \boldsymbol{b}$ and $Spec\ \boldsymbol{a} \subseteq Spec\ \boldsymbol{b}$.

As equivalence is a symmetric relation, we have $Spec\ \boldsymbol{a} = Spec\ \boldsymbol{b}$.

In a similar way, if for the subsequence $\boldsymbol{d} = (d_i)_{i \in \omega}$,

$$\lim_{i\to\infty} d_i = \infty$$

then the same is true for the subsequence $\boldsymbol{c} = (c_i)_{i \in \omega}$, i.e.,

$$\lim_{i\to\infty} c_i = \infty$$

and if

$$\lim_{i\to\infty} d_i = -\infty$$

the same is true for the sequence $\boldsymbol{c} = (c_i)_{i\in\omega}$,

$$\lim_{i\to\infty} c_i = -\infty$$

Thus, E*Spec* $\boldsymbol{a} \subseteq$ E*Spec* $\boldsymbol{b}$ and as equivalence is a symmetric relation, E*Spec* $\boldsymbol{a} =$ E*Spec* $\boldsymbol{b}$.

Lemma is proved.

This result allows us to define spectrum and extended spectrum for hypernumbers.

Definition 2.1.12

(a) The *spectrum Spec* α of a real hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ is equal to the spectrum *Spec* $\boldsymbol{a}$ of its defining sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$, i.e., *Spec* $\alpha = \{r \in \boldsymbol{R}; r = \lim_{i\in C} a_i$ and C is an infinite subset of $\omega\}$.

(b) The *extended spectrum* E*Spec* α of a real hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ is equal to the extended spectrum *Spec* $\boldsymbol{a}$ of its defining sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$.

In other words, the spectrum of a hypernumber is the set of all limit points of one of its representations and by Lemma 2.1.9, the spectrum and the extended spectrum of a hypernumber do not depend on the choice of a sequence $(a_i)_{i\in\omega}$ that represents this hypernumber.

Example 2.1.10 If $a = (a_i)_{i\in\omega}$ and $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = i$, $i = 1, 2, 3, \ldots$, then *Spec* $\alpha =$ *Spec* $\boldsymbol{a} = \varnothing$ and E*Spec* $\alpha =$ E*Spec* $\boldsymbol{a} = \{\infty\}$.

Example 2.1.11 If $\boldsymbol{a} = (a_i)_{i\in\omega}$ and $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = (-1)^i$, $i = 1, 2, 3, \ldots$, then *Spec* $\alpha =$ E*Spec* $\alpha =$ *Spec* $\boldsymbol{a} = \{1, -1\}$.

Example 2.1.12 If $\boldsymbol{a} = (a_i)_{i\in\omega}$ and $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = (-1)^i \cdot i$, $i = 1, 2, 3, \ldots$, then *Spec* $\boldsymbol{a} =$ *Spec* $\alpha = \varnothing$ and E*Spec* $\alpha =$ E*Spec* $\boldsymbol{a} = \{\infty, -\infty\}$.

Example 2.1.13 If $\boldsymbol{a} = (a_i)_{i\in\omega}$ and $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = 1/i$, $i = 1, 2, 3, \ldots$, then *Spec* $\alpha =$ E*Spec* $\alpha =$ *Spec* $\boldsymbol{a} = \{0\}$.

Example 2.1.14 If $\boldsymbol{a} = (a_i)_{i\in\omega}$ and $a = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_{2i} = m/n$ and $a_{2i-1} = -m/n$ if $i = [(m + n)^2 + 3m + n]/2$, and $m, n = 1, 2, 3, \ldots$, then *Spec* $\alpha =$ *Spec* $\boldsymbol{a} = \boldsymbol{R}$ and E*Spec* $\alpha =$ E*Spec* $\boldsymbol{a} = \boldsymbol{R} \bigcup \{\infty, -\infty\}$ because the mapping $\mathbf{dc}(m, n) = [(m + n)^2 + 3m + n]/2$ is the Cantor enumeration, which is a one-to-one correspondence between all pairs of natural numbers and all natural numbers. It means that any rational number is equal to one of the elements a_i.

Example 2.1.15 If $\boldsymbol{a} = (a_i)_{i\in\omega}$ and $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = m/n - [m/n]$ if $i = [(m + n)^2 + 3m + n]/2$, and m, $n = 1$, 2, 3, …, then *Spec* $\alpha =$ *Spec* $\boldsymbol{a} =$ E*Spec* $\alpha =$ E*Spec* $\boldsymbol{a} = [0, 1]$ because any rational number that is larger than or equal to 0 and less than 1 is equal to one of the elements a_i.

Proposition 2.1.6 *For any hypernumber α, Spec α is a closed set.*

Proof Let us take a hypernumber $\alpha = \text{Hn}(a_i)_{i\in\omega}$ and assume that *Spec* α contains elements r_n such that for each $i = 1, 2, 3, \ldots,\ r_n = \lim_{i\to\infty} b_{n,i}$ for some subsequence $(b_{n,i})_{i\in\omega}$ of the sequence $(a_i)_{i\in\omega}$ and $r = \lim_{n\to\infty} r_n$. By the definition of a limit, it is possible to choose an element $b_{n,i(n)}$ from the sequence $(b_{n,i})_{i\in\omega}$ such that $|r_n - b_{n,i(n)}| < 1/n$. Taking the sequence $(b_{n,i(n)})_{n\in\omega}$, we see that it is a subsequence of the sequence $(a_i)_{i\in\omega}$ and $r = \lim_{n\to\infty} b_{n,i(n)}$. Thus, r belongs to *Spec* α and *Spec* α is a closed set by the definition of topology in $\boldsymbol{R}$.

Proposition is proved because α is an arbitrary hypernumber.

The spectrum of a hypernumber is a characteristic that allows one to discern different types of hypernumbers. The definitions imply the following results.

Proposition 2.1.7 $\alpha = \text{Hn}(a_i)_{i\in\omega}$ *is a bounded oscillating real hypernumber if and only if* E*Spec* α = *Spec* α *contains more than one element.*

Proof is left as an exercise.

Proposition 2.1.8 *A hypernumber* $\alpha = \text{Hn}(a_i)_{i\in\omega}$ *is a real number if and only if* ESpec α *is equal to* Spec α *and contains only one element.*

Proof is left as an exercise.

Proposition 2.1.9 *Two stable real hypernumbers α and β are equal if and only if Spec* α = *Spec* β = E*Spec* α = E*Spec* β.

Indeed, any stable real hypernumber α is a real number, say, a, and *Spec* α = E*Spec* $\alpha = \{a\}$. Thus, equality of spectra implies equality of hypernumbers.

For finite real hypernumbers, this result is not true in a general case. Indeed, if $\alpha = \text{Hn}(a_i)_{i\in\omega}$ and $\beta = \text{Hn}(b_i)_{i\in\omega}$ where $a_i = (-1)^i$ *and* $b_i = (-1)^{i+1}$, i = 1, 2, 3, …, *then* $a \neq \beta$ but *Spec* α = *Spec* β = E*Spec* α = E*Spec* $\beta = \{1, -1\}$.

Proposition 2.1.10 *The spectrum Spec α is empty if and only if α is either an infinite increasing or infinite decreasing or strictly infinite expanding hypernumber.*

Indeed, in all these cases and only in these cases, representing sequences of hypernumbers do not have converging subsequences.

2.2 Algebraic Properties of Hypernumbers

Relations on $\boldsymbol{R}$ *induce corresponding relations on* $\boldsymbol{R}^\omega$ *through pairwise relations between members of representing sequences with the same index number.*

Definition 2.2.1 For any $\boldsymbol{a} = (a_i)_{i\in\omega}, \boldsymbol{b} = (b_i)_{i\in\omega} \in \boldsymbol{R}^\omega$:

$$\boldsymbol{a} \leq \boldsymbol{b} \text{ if } \exists n \in \omega\ \forall i(a_i \leq b_i)$$

and

$$\boldsymbol{a} < \boldsymbol{b} \text{ if } \exists\, n \in \omega\ \forall i (a_i < b_i)$$

Lemma 2.2.1 (Burgin 2002) *Relations* $\leq$ *and* $<$ in $\boldsymbol{R}^{\omega}$ *are a partial preorder and a strict partial order, respectively.*

These relations induce similar relations on $\boldsymbol{R}_{\omega}$.

Definition 2.2.2 If $\alpha, \beta \in \boldsymbol{R}_{\omega}$, then

$$\alpha \leq \beta \text{ means that } \exists\, \boldsymbol{a} \in \alpha\ \exists\, \boldsymbol{b} \in \beta (\boldsymbol{a} \leq \boldsymbol{b})$$

$$\alpha < \beta \text{ means that } (\exists\, \boldsymbol{a} \in \alpha\ \exists\, \boldsymbol{b} \in \beta (\boldsymbol{a} < \boldsymbol{b})) \quad \text{and} \quad \alpha \neq \beta$$

If we take hypernumbers $\alpha, \beta, \gamma, \delta, \theta$, and ν from the Examples 2.1.2–2.1.7, then we have the following relations: $\alpha < \beta, \alpha > \nu, \gamma < \alpha, \gamma < \beta, \delta < \theta, \alpha < \theta, \beta > \delta, \alpha \geq \delta$ and $\nu < \gamma$, while hypernumbers γ and δ as well as hypernumbers β and θ, are incomparable with respect to the order relations. This shows that real hypernumbers may be incomparable in contrast to real numbers, where for any two numbers a and b, we have either $a = b$ or $a > b$ or $a < b$.

Lemma 2.2.2

(a) $\alpha \leq \beta$ *if and only if there are a sequence* $\boldsymbol{a} = (a_i)_{i \in \omega}$ *that represents* α *and a sequence* $\boldsymbol{b} = (b_i)_{i \in \omega}$ *that represents* β *such that* $a_i \leq b_i$ *for all* $i = 1, 2, 3, \ldots$

(b) $\alpha < \beta$ *if and only if there are a sequence* $\boldsymbol{a} = (a_i)_{i \in \omega}$ *that represents* α *and a sequence* $\boldsymbol{b} = (b_i)_{i \in \omega}$ *that represents* β *such that* $k < b_i - a_i$ *for some positive real number* k *and all* $i = 1, 2, 3, \ldots$

Proof is left as an exercise.

Lemma 2.2.3 *Relations* $\leq$ *and* $<$ *in* $\boldsymbol{R}_{\omega}$ *are a partial order and a strict partial order, respectively.*

Proof By the definition, a strict partial order is a transitive asymmetric relation and a partial order is a reflexive transitive antisymmetric relation (cf. Appendix). Thus, we have to test these properties.

1. *Transitivity*. Let us assume $\alpha < \beta$ and $\beta < \gamma$ for some hypernumbers $\alpha, \beta, \gamma \in \boldsymbol{R}_{\omega}$. By the definition of the relation $<$, there are such sequences $\boldsymbol{a} = (a_i)_{i \in \omega} \in \alpha$, $\boldsymbol{b} = (b_i)_{i \in \omega}$, $\boldsymbol{l} = (l_i)_{i \in \omega} \in \beta$, and $\boldsymbol{c} = (c_i)_{i \in \omega} \in \gamma$, for which the following conditions are valid: there is a number n such that any inequality $i > n$ implies $a_i < b_i$, and there is a number m such that any inequality $j > m$ implies $l_j < c_j$. In addition, $\lim_{i \to \infty}(b_i - a_i) \neq 0$ and $\lim_{i \to \infty}(c_i - l_i) \neq 0$. Then there are positive numbers k and $h \in \boldsymbol{R}^{++}$ such that $b_i - a_i > k$ for all $i > n$ and $c_j - l_j > h$ for all $j > m$.

   Let $g = \min\{k, h\}$. Then there is a natural number p such that $|b_j - l_j| < g$ when $i > p$ because the sequences $\boldsymbol{b} = (b_i)_{i \in \omega}$ and $\boldsymbol{l} = (l_i)_{i \in \omega}$ define the same real hypernumber β.

Let $q = \max\{m, n, p\}$. Then assumming $i > q$, we have

$$\begin{aligned} c_i - a_i = c_i - b_i + b_i - a_i &= c_i - l_i + l_i - b_i + b_i - a_i \\ &= (c_i - l_i) + (l_i - b_i) + (b_i - a_i) > g - g + g = g \end{aligned}$$

because $b_i - a_i > g$, $c_i - l_i > g$ and $l_j - b_j > -g$ because $|b_i - l_i| < g$. Thus, $c_i > a_i$ when $i > q$ and $\lim_{i\to\infty}(c_i - a_i) \neq 0$. By the definition of the relation $<$, we have $\alpha < \gamma$. As α, β, γ are arbitrary points (hypernumbers) from $\boldsymbol{R}_\omega$, it means that the relation $<$ is transitive.

The case when $\alpha \leq \beta$ and $\beta \leq \gamma$ for some hypernumbers $\alpha, \beta, \gamma \in \boldsymbol{R}_\omega$ is treated in a similar way, demonstrating that the relation $\leq$ is also transitive.

2. *Asymmetry/antisymmetry*. Now we show that the relation $<$ is asymmetric and relation $\leq$ is antisymmetric in $\boldsymbol{R}_\omega$. Let us suppose that this is not true for the relation $>$. It means that for some real hypernumbers $\alpha, \beta \in \boldsymbol{R}_\omega$, we have $\alpha < \beta$ and $\beta < \alpha$. Then by the definition of the relation $<$, $\alpha \neq \beta$ and there are such sequences $\boldsymbol{a} = (a_i)_{i\in\omega}$, $\boldsymbol{d} = (d_i)_{i\in\omega} \in \alpha$ and $\boldsymbol{b} = (b_i)_{i\in\omega}$, $\boldsymbol{l} = (l_i)_{i\in\omega} \in \beta$, for which $a_i < b_i$ and $l_i < l_i$ for all $i = 1, 2, 3, \ldots$. Consequently, $\lim_{i\to\infty}(b_i - a_i) \geq 0$. At the same time, we have

$$\begin{aligned} \lim_{i\to\infty}(b_i - a_i) &= \lim_{i\to\infty}(b_i - l_i + l_i - a_i) \\ &= \lim_{i\to\infty}((b_i - l_i) + (l_i - d_i) + (d_i - a_i)) \\ &= \lim_{i\to\infty}(b_i - l_i) + \lim_{i\to\infty}(l_i - d_i) + \lim_{i\to\infty}(d_i - a_i) \\ &= \lim_{i\to\infty}(l_i - d_i) \end{aligned}$$

because $\lim_{i\to\infty}(b_i - l_i) = 0$ and $\lim_{i\to\infty}(b_i - a_i) = 0$. However, $\lim_{i\to\infty}(l_i - d_i) \leq 0$ because $l_i < d_i$ for all $I = 1, 2, 3, \ldots$. Consequently, $\lim_{i\to\infty}(b_i - a_i) = \lim_{i\to\infty}(\mathrm{l_i} - \mathrm{d_i}) = 0$. It means, by the definition of a hypernumber, that $\alpha = \beta$. This contradicts our assumptions and proves that the relation $<$ is asymmetric on $\boldsymbol{R}_\omega$ by the Law of Contraposition for propositions (cf. Church 1956).

Taking the relation $\leq$, we see that the above proof shows that if we have $\alpha \leq \beta$ and $\beta \leq \alpha$, then $\alpha = \beta$, i.e., the relation $\leq$ is antisymmetric. In addition to this, by the definition of a real hypernumber, we have that $\alpha \leq \alpha$, i.e., the relation $\leq$ is reflexive. Thus, $\leq$ is a partial order in $\boldsymbol{R}_\omega$.

Lemma is proved.

This allows us to define:

- The set $\boldsymbol{R}_\omega^{++}$ of all positive real hypernumbers
- The set $\boldsymbol{R}_\omega^{+}$ of all non-negative real hypernumbers
- The set $\boldsymbol{R}_\omega^{-}$ of all non-positive real hypernumbers
- The set $\boldsymbol{R}_\omega^{--}$ of all negative real hypernumbers

It is interesting that although real numbers are isomorphically included into the set of all real hypernumbers, some concepts change their meaning. One of them is the concept of an interval. As we know an interval in $\boldsymbol{R}$ is uniquely determined by its endpoints. The same is true for an interval in $\boldsymbol{R}_\omega$. However, the same endpoints determine different intervals in $\boldsymbol{R}$ and in $\boldsymbol{R}_\omega$. For instance, the interval $[-1, 1]$ in $\boldsymbol{R}$ contains only real numbers, while the interval $[-1, 1]$ in $\boldsymbol{R}_\omega$ also contains proper real hypernumbers, such as $\gamma = \mathrm{Hn}(a_i)_{i\in\omega}$ where $a_i = (-1)^i$, $i = 1, 2, 3, \ldots$ or $\beta = \mathrm{Hn}(b_i)_{i\in\omega}$ where $b_i = \frac{1}{2}$ for $i = 2, 4, \ldots, 2n, \ldots$ and $b_i = \frac{1}{4}$ for $i = 1, 3, \ldots, 2n-1, \ldots$.

Operations in $\boldsymbol{R}$ induce corresponding operations in $\boldsymbol{R}^\omega$.

Let $\boldsymbol{a} = (a_i)_{i\in\omega}$ and $\boldsymbol{b} = (b_i)_{i\in\omega}$ are elements from $\boldsymbol{R}^\omega$. Then we have the following constructions.

Definition 2.2.3

(a) Operation of *addition* in $\boldsymbol{R}^\omega$ is defined as $\boldsymbol{a} + \boldsymbol{b} = (c_i)_{i\in\omega}$ where $c_i = a_i + b_i$ for all $i \in \omega$

(b) Operation of *subtraction* in $\boldsymbol{R}^\omega$ is defined as $\boldsymbol{a} - \boldsymbol{b} = (c_i)_{i\in\omega}$ where $c_i = a_i - b_i$ for all $i \in \omega$

(c) Operation of *multiplication* in $\boldsymbol{R}^\omega$ is defined as $\boldsymbol{a} \cdot \boldsymbol{b} = (c_i)_{i\in\omega}$ where $c_i = a_i \cdot b_i$ for all $i \in \omega$

(d) Operation of division in $\boldsymbol{R}^\omega$ is defined as $\boldsymbol{a}/\boldsymbol{b} = (c_i)_{i\in\omega}$ where $c_i = a_i/b_i$ and $b_i \neq 0$for all $i \in \omega$

(e) Operation of taking the *maximum* in $\boldsymbol{R}^\omega$ is defined as $\max\{\boldsymbol{a}, \boldsymbol{b}\} = (c_i)_{i\in\omega}$ where $c_i = \max\{a_i, b_i\}$ for all$i \in \omega$

(f) Operation of taking the *minimum* in $\boldsymbol{R}^\omega$ is defined as $\min\{\boldsymbol{a}, \boldsymbol{b}\} = (c_i)_{i\in\omega}$ where $c_i = \min\{a_i, b_i\}$ for all $i \in \omega$

Remark 2.2.1 By the definition of addition and multiplication in the set $\boldsymbol{R}^\omega$, all laws of operations with real numbers (commutativity of addition, associativity of addition, commutativity of multiplication, associativity of multiplication, and distributivity) are valid for corresponding operations with sequences of real numbers.

Proposition 2.2.1 *Operations of min and max in* $\boldsymbol{R}^\omega$ *induce similar operations in* $\boldsymbol{R}_\omega$.

Proof is left as an exercise.

Proposition 2.2.2 *Operations of addition and subtraction in* $\boldsymbol{R}^\omega$ *induce similar operations in* $\boldsymbol{R}_\omega$.

Proof Let us take some real hypernumbers $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ and $\beta = \mathrm{Hn}(b_i)_{i\in\omega}$. We define $\alpha + \beta = \gamma$ from $\boldsymbol{R}_\omega$, where $\gamma = \mathrm{Hn}(a_i + b_i)_{i\in\omega}$. To show that this is an operation in $\boldsymbol{R}_\omega$, it is necessary to prove that γ does not depend on the choice of sequences a and b. To do this, let us take another sequence $\boldsymbol{l} = (l_i)_{i\in\omega}$ in β and show that if the real hypernumber δ is equal to $\mathrm{Hn}(a_i + l_i)_{i\in\omega}$, then $\delta = \gamma$.

By Definition 2.1.3, we have $\lim_{i\to\infty}|b_i - l_i| = 0$. Consequently, $\lim_{i\to\infty}|(a_i + b_i) - (a_i + l_i)| = \lim_{i\to\infty}|b_i - l_i| = 0$. Then by Definition 2.1.3, $\delta = \gamma$.

If we take another real sequence that represents the hypernumber α, a similar proof shows that the result of addition $\alpha + \beta$ will be the same.

The proof for the difference of two hypernumbers is similar to the proof for the sum of two hypernumbers.

Proposition is proved.

Proposition 2.2.3 *If* $\alpha \leq \beta$ *and* γ*, then* $\alpha + \gamma \leq \beta + \gamma$*, i.e., operation of addition is monotone.*

Indeed, $\alpha \leq \beta$ implies existence sequences of real numbers $\boldsymbol{a} = (a_i)_{i \in \omega}$ and $\boldsymbol{b} = (b_i)_{i \in \omega}$ such that $\alpha = \mathrm{Hn}(a_i)_{i \in \omega}$, $\beta = \mathrm{Hn}(b_i)_{i \in \omega}$ and

$$\boldsymbol{a} \leq \boldsymbol{b} \text{ if } \exists\, n \in \omega\ \forall i (a_i \leq b_i)$$

Thus, if $\gamma = \mathrm{Hn}(c_i)_{i \in \omega}$, then for almost all i, we have $a_i + c_i \leq b_i + c_i$ and by Definition 2.2.1, $\alpha + \gamma \leq \beta + \gamma$.

Theorem 2.2.1 *The operation of multiplication is defined in* $\boldsymbol{FR}_\omega$*.*

Proof Let us take an arbitrary finite real hypernumber $\beta = \mathrm{Hn}(b_i)_{i \in \omega} \in \boldsymbol{R}_\omega$ and an arbitrary finite real hypernumber $\alpha = \mathrm{Hn}(a_i)_{i \in \omega} \in \boldsymbol{R}_\omega$. Then we define $\boldsymbol{a} \cdot \boldsymbol{b} = (c_i)_{i \in \omega}$ where $\boldsymbol{a} = (a_i)_{i \in \omega}$, $\boldsymbol{b} = (b_i)_{i \in \omega}$, $c_i = a_i \cdot b_i$ for all $i \in \omega$. We define the product of α and β as $\alpha \cdot \beta = \gamma = \mathrm{Hn}(a_i \cdot b_i)_{i \in \omega}$. To show that this is an operation in $\boldsymbol{R}_\omega$, it is necessary to prove that γ does not depend on the choice of the sequences $\boldsymbol{a}$ and $\boldsymbol{b}$. To do this, let us take another sequence $\boldsymbol{l} = (l_i)_{i \in \omega}$ in β and show that if the real hypernumber δ is equal to $\mathrm{Hn}(a_i \cdot l_i)_{i \in \omega}$, then $\delta = \gamma$.

By Definition 2.1.3, $\lim_{i \to \infty} |b_i - l_i| = 0$. Besides, there is a number $c \in \boldsymbol{R}^+$ such that $|a_i| \leq c$ for all $i \in \omega$. Consequently, we have

$$\lim_{i \to \infty} |a_i \cdot b_i - a_i \cdot l_i| = \lim_{i \to \infty} |a_i| \cdot |b_i - l_i| \leq \lim_{i \to \infty} |c| \cdot |b_i - l_i| = 0$$

Then by Definition 2.1.3, $\alpha \cdot \beta = \mathrm{Hn}(a_i \cdot l_i)_{i \in \omega}$. It means that the definition of the product $\alpha{\cdot}\beta$ does not depend on the choice of a sequence from β. Similar arguments show that the same is true for the hypernumber α. Thus, multiplication is correctly defined in $\mathbf{F}\boldsymbol{R}_\omega$.

Theorem is proved.

Remark 2.2.2 By the definition of addition and multiplication in the set $\boldsymbol{R}_\omega$, all laws of operations with real numbers (commutativity of addition, associativity of addition, commutativity of multiplication, associativity of multiplication, and distributivity) are valid for corresponding operations with bounded real hypernumbers.

As any stable sequence is bounded, we have the following result.

Corollary 2.2.1 *The operation of multiplication by stable real sequences induces the operation of multiplication of real hypernumbers by elements of* $\boldsymbol{R}$*, i.e., by real numbers.*

Properties of limits imply the following result.

Proposition 2.2.4 *For any real number* a *and any real hypernumber* α*, we have*

$$Spec\ a\alpha = \{ac; c \in Spec\ \alpha\}$$

However, in a general case, a similar result is not true for addition and subtraction. For instance, it is possible that

$$Spec(\alpha + \beta) \neq \{a + c; a \in Spec\ \alpha \text{ and } c \in Spec\ \beta\}$$

It is even possible that $Spec\ \alpha \neq \varnothing$ and $Spec\ \beta\ \neq \varnothing$ but $Spec(\alpha + \beta) = \varnothing$. Indeed, let us consider hypernumbers $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ and $\beta = \mathrm{Hn}(b_i)_{i\in\omega}$ where

$$a_i = \begin{cases} k & i = 2k \\ 1 & \text{if } i = 2k - 1 \end{cases}$$

$$b_i = \begin{cases} k & \text{if } i = 2k - 1 \\ 1 & \text{if } i = 2k \end{cases}$$

In this case, Spec α = Spec β = *{1}* but $Spec(\alpha + \beta) = \varnothing$.

Definition 2.2.4 A sequence $(a_i)_{i\in\omega}$ of real numbers is *separated from* a real number a if $|a - a_i| > k > 0$ for some number k and all $i \in \omega$.

For instance, a sequence $(a_i)_{i\in\omega}$ of real numbers is *separated from* 0 if $|a_i| > k$ for some number $k > 0$ and all $i \in \omega$.

Definition 2.2.5 A hypernumber α is separated from a real number a if there is a sequence $(a_i)_{i\in\omega}$ of real numbers separated from a such that $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$.

For instance, a hypernumber α is *separated from* 0 if there is a separated from 0 sequence $(a_i)_{i\in\omega}$of real numbers such that $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$.

We denote by $\mathbf{S}\boldsymbol{R}_\omega$ the set of all real hypernumbers separated from 0 and by $\mathbf{SB}\boldsymbol{R}_\omega$ the set that consists of the hypernumber 0 and of all bounded real hypernumbers that are separated from 0.

Lemma 2.2.4 *A hypernumber α is separated from* 0 *if and only if* 0 *does not belong to its spectrum.*

Proof
Necessity. Let us assume that a hypernumber α is separated from 0. Then $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ where the sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ is separated from 0. Consequently, 0 cannot be a limit of any subsequence of the sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$. It means that 0 does not belong to the spectrum *Spec* $\boldsymbol{a}$. At the same time, by Definition 2.1.12, $Spec\ \alpha = Spec\ \boldsymbol{a} = (a_i)_{i\in\omega}$. So, 0 does not belong to the spectrum *Spec* α.

Sufficiency. Let us assume that a hypernumber α is not separated from 0. It means that if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$, then the sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ is not separated from 0. So, by Definition 2.2.4, for any natural number n, there is an element a_i such that $a_i < 1/n$. This gives us a subsequence of that converges to 0. Consequently, by Definitions 2.1.11 and 2.1.12, $0 \in Spec\ \boldsymbol{a} = Spec\ \alpha$. Thus, by the Law of Contraposition for propositions (cf. Church 1956), if 0 does not belong to the spectrum *Spec* α, then the hypernumber α is separated from 0.

Lemma is proved.

As the product of two separated from 0 real sequences is also a separated from 0 real sequence, we have the following result.

Lemma 2.2.5 $\boldsymbol{SBR}_\omega$ *is a linear algebra.*

Definition 2.2.6 If all $b_i \neq 0$, then the sequence $(a_i/b_i)_{i\in\omega}$ of real numbers defines the *sequential division* of sequences $\boldsymbol{a} = (a_i)_{i\in\omega}$ and $\boldsymbol{b} = (b_i)_{i\in\omega}$ of real numbers.

Sequential division of real sequences allows us to define sequential division of real hypernumbers.

Theorem 2.2.2 *The sequential division of sequences correctly defines division of bounded hypernumbers by hypernumbers separated from* 0.

Proof Let us take an arbitrary bounded real hypernumber $\beta \in \mathbf{F}\boldsymbol{R}_\omega$ and an arbitrary finite real hypernumber $\alpha \in \boldsymbol{R}_\omega$, which is separated from 0. By Definition 2.2.3, there is a sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ of real numbers such that $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$and $|a_i| > k$ for some number $k > 0$ and all $i \in \omega$.

Then we define $\boldsymbol{b}/\boldsymbol{a} = (c_i)_{i\in\omega}$ where $c_i = b_i/a_i$ for all $i \in \omega$. Taking a real sequence $b = (b_i)_{i\in\omega}$ such that $\beta = \mathrm{Hn}(b_i)_{i\in\omega}$, we define $\beta/\alpha = \gamma$ from $\boldsymbol{R}_\omega$, where $\gamma = \mathrm{Hn}(b_i/a_i)_{i\in\omega}$. The real sequence $(b_i/a_i)_{i\in\omega}$ is correctly defined because all numbers a_i are not equal to 0. Thus, real hypernumber is also correctly defined.

To show that this is an operation in $\boldsymbol{R}_\omega$, it is necessary to prove that γ does not depend on the choice of the real sequences $\boldsymbol{a}$ and $\boldsymbol{b}$. To do this, let us, at first, take another sequence $\boldsymbol{l} = (l_i)_{i\in\omega}$ that represents β and show that if the real hypernumber δ is equal to $\mathrm{Hn}(l_i/a_i)_{i\in\omega}$, then $\delta = \gamma$.

By definition, $\lim_{i\to\infty}|b_i - l_i| = 0$. Consequently, we have

$$\begin{aligned}\lim_{i\to\infty}|(b_i/a_i) - (l_i/a_i)| &= \lim_{i\to\infty}|b_i - l_i|/|a_i| \leq \lim_{i\to\infty}(|b_i - l_i|/|k|)\\ &= \left(\lim_{i\to\infty}|b_i - l_i|\right)/|k| = 0\end{aligned}$$

Then by Definition 2.1.3, $\beta/\alpha = \mathrm{Hn}(l_i/a_i)_{i\in\omega}$. It means that the definition of the ratio β/α does not depend on the choice of a sequence that represents β.

Now let us, at first, take another separated from 0 sequence $t = (t_i)_{i\in\omega}$ in α and show that if the real hypernumber δ is equal to $\mathrm{Hn}(b_i/t_i)_{i\in\omega}$, then $\delta = \gamma$.

By definition, $\lim_{i\to\infty}|a_i - t_i| = 0$. Besides, as the hypernumber β is bounded, there is such $c \in \boldsymbol{R}^+$ that $|b_i|$ for all $i = 1, 2, 3, \ldots$. Consequently, we have

$$\begin{aligned}\lim_{i\to\infty}|(b_i/a_i) - (b_i/t_i)| &= \lim_{i\to\infty}(|b_i| \cdot |t_i - a_i/(a_i t_i)|)\\ &= \lim_{i\to\infty}(|b_i|/|a_i t_i|) \cdot |t_i - a_i|\\ &\leq \lim_{i\to\infty}(|b_i|/k^2) \cdot |t_i - a_i| \leq \lim_{i\to\infty}(c/k^2) \cdot |t_i - a_i|\\ &= (c/k^2)\lim_{i\to\infty}|t_i - a_i| = 0\end{aligned}$$

Then by Definition 2.1.3, $\beta/\alpha = \mathrm{Hn}(l_i/a_i)_{i\in\omega}$. It means that the definition of the ratio β/α does not depend on the choice of a sequence from α.

Consequently, division of bounded hypernumbers by hypernumbers separated from 0 is correctly defined.

Theorem is proved.

Corollary 2.2.2 *Any element from* $\boldsymbol{SR}_\omega$ *has an inverse element in* $\boldsymbol{R}_\omega$.

However, such inverse element must not belong to $\boldsymbol{SR}_\omega$.

Thus, we have demonstrated that operations in $\boldsymbol{R}^{\omega}$ induce similar operations in $\boldsymbol{R}_{\omega}$. Namely, if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ and $\beta = \mathrm{Hn}(b_i)_{i\in\omega}$, then

$$\alpha + \beta = \mathrm{Hn}(a_i + b_i)_{i\in\omega}$$

$$\alpha - \beta = \mathrm{Hn}(a_i - b_i)_{i\in\omega}$$

$\alpha \cdot \beta = \mathrm{Hn}(a_i \cdot b_i)_{i\in\omega}$ when all numbers a_i or all numbers b_i are bounded
$\alpha/\beta = \mathrm{Hn}(a_i/b_i)_{i\in\omega}$ when there is $k > 0$, such that $|b_i| > k$ for all $i \in \omega$

$$\max\{\alpha = \mathrm{Hn}(a_i)_{i\in\omega}, \beta = \mathrm{Hn}(b_i)_{i\in\omega}\} = \mathrm{Hn}(\max\{a_i, b_i\})_{i\in\omega}$$

$$\min\{\alpha = \mathrm{Hn}(a_i)_{i\in\omega}, \beta = \mathrm{Hn}(b_i)_{i\in\omega}\} = \mathrm{Hn}(\min\{a_i, b_i\})_{i\in\omega}$$

Theorem 2.2.3 *The set $\boldsymbol{R}_\omega$ of all real hypernumbers is an ordered linear infinite dimensional space over the field **R** of real numbers, in which the binary operations max and min are defined.*

Proof By Proposition 2.2.2 and Corollary 2.2.1, the set $\boldsymbol{R}_\omega$ of all real hypernumbers is a linear space over the field $\boldsymbol{R}$. Let us consider real hypernumbers $\alpha^t = \mathrm{Hn}(a_i{}^t)_{i\in\omega}$ with $t \in \omega$ and $a_i^t = (i)^t$ for all $t, i \in \omega$. We can see that these elements are linearly independent in $\boldsymbol{R}_\omega$. Consequently, the space $\boldsymbol{R}_\omega$ is infinite dimensional.

The set $\boldsymbol{R}_\omega$ is partially ordered. So, to prove the theorem, we have only to demonstrate that this order is compatible with operations in $\boldsymbol{R}_\omega$ (Fuchs 1963). In the case of linear spaces, it means that we have to show that $\alpha \leq \beta$ implies $a\alpha\beta$ for any $\alpha, \beta \in \boldsymbol{R}_\omega$ and $a \in \boldsymbol{R}^+$, and it implies $\alpha + \gamma \leq \beta + \gamma$ for any $\gamma \in R_\omega$.

Let $\alpha \leq \beta$ for some hypernumbers $\alpha, \beta \in \boldsymbol{R}_\omega$. By the definition of the relation $\leq$, there are such sequences $\boldsymbol{a} = (a_i)_{i\in\omega} \in \alpha$ and $\boldsymbol{b} = (b_i)_{i\in\omega} \in \beta$, for which the following conditions are valid: there is a natural number n such that if $i > n$, then $a_i \leq b_i$. Then for any $a \in \boldsymbol{R}^+$, by the properties of real numbers, we have aa_ib_i for all $i > n$. Consequently, $a\alpha\beta$.

Let $\gamma = \mathrm{Hn}(c_i)_{i\in\omega} \in \boldsymbol{R}_\omega$ and $\alpha \leq \beta$. By the definition of the relation $\leq$, there are such sequences $\boldsymbol{a} = (a_i)_{i\in\omega}$ that represents α and $\boldsymbol{b} = (b_i)_{i\in\omega}$ that represents β, for which the following conditions are valid: there is a natural number n such that if $i > n$, then $a_i \leq b_i$. Then $\alpha + \gamma = \mathrm{Hn}(a_i + c_i)_{i\in\omega}$ and $\alpha + \beta = \mathrm{Hn}(b_i + c_i)_{i\in\omega}$. By the properties of real numbers, we have $a_i + c_i \leq b_i + c_i$ for all $i > n$. This implies $\alpha + \gamma \leq \beta + \gamma$.

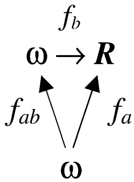
Theorem is proved.

Corollary 2.2.3 *The set $\boldsymbol{FR}_\omega$ of all finite real hypernumbers is an ordered algebra over the field **R** of real numbers.*

Corollary 2.2.4 *The set $\boldsymbol{SBR}_\omega$ that consists of the hypernumber* 0 *and all finite real hypernumbers that are separated from zero is an ordered field and is an ordered algebra over the field **R** of real numbers.*

Relations between and operations with real hypernumbers studied above are extensions of similar relations and operations that exist in the space $\boldsymbol{R}$ of all real numbers. At the same time, there are relations between and operations with real hypernumbers such that they have no counterparts in the space $\boldsymbol{R}$. A more general nature of real hypernumbers, in comparison with real numbers, brings us to new constructions related to hypernumbers. One of such unusual relations is the relation "to be a subhypernumber."

Definition 2.2.7 A sequence $(a_i)_{i\in\omega}$ is a *subsequence* of a sequence $(b_i)_{i\in\omega}$ if there is a strictly increasing function $f_{ab} : \omega \to \omega$ such that for any $i \in \omega$, we have $a_i = b_{f_{ab}(i)}$, i.e., the following diagram is commutative



The concept of a subsequence brings us to the concept of a subhypernumber.

Definition 2.2.8 (a) A real hypernumber α is called a *subhypernumber* of a real hypernumber β if there are sequences $\boldsymbol{a} = (a_i)_{i\in\omega}$ and $\boldsymbol{b} = (b_i)_{i\in\omega}$ such that $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$, $\beta = \mathrm{Hn}(b_i)_{i\in\omega}$, and $(a_i)_{i\in\omega}$ is a subsequence of the sequence $(b_i)_{i\in\omega}$. (b) If α is a subhypernumber of β and $\alpha \neq \beta$, then α is called a *proper subhypernumber* of β.

We denote this relation by $\alpha \Subset \beta$ and $\alpha \subset \beta$, correspondingly. Sub $\alpha = \{\gamma: \gamma \Subset \alpha\}$ denotes the set of all subhypernumbers of the real hypernumber α. If $\alpha \Subset \beta$, then β is called a *superhypernumber* of α.

Note that real numbers do not have proper subhypernumbers.

Lemma 2.2.6 *If* $\alpha \Subset \beta$ *and* $\beta \in \boldsymbol{R}$, *then* $\alpha \in \boldsymbol{R}$ *and* $\alpha = \beta$.

Proof *is left as an exercise.*

The concept of a subhypernumber is defined using representatives of hypernumbers. However, it does not depend on the choice of a representing sequence of its superhypernumber as is implied by the following result.

Lemma 2.2.7 *If* $\beta = \mathrm{Hn}(b_i)_{i\in\omega} = \mathrm{Hn}(c_i)_{i\in\omega}$ *and* $\alpha = \mathrm{Hn}(a_j)_{j\in\omega}$ *where* $(a_j)_{j\in\omega}$ *is a subsequence of* $(b_i)_{i\in\omega}$, *then there is a subsequence* $(d_j)_{j\in\omega}$ *of* $(c_i)_{i\in\omega}$ *such that* $\alpha = \mathrm{Hn}(d_j)_{j\in\omega}$.

Indeed, to build the necessary subsequence of the sequence $(c_i)_{i\in\omega}$, we can take elements d_j from the sequence $(c_i)_{i\in\omega}$ that have the same indices as elements a_j in the sequence $(b_i)_{i\in\omega}$. As $\mathrm{Hn}(b_i)_{i\in\omega} = \mathrm{Hn}(c_i)_{i\in\omega}$, we have $\lim_{i\to\infty}|b_i - c_i| = 0$. Consequently, $\lim_{i\to\infty}|a_i - d_i| = 0$. Thus, $\mathrm{Hn}(a_j)_{j\in\omega} = \mathrm{Hn}(d_j)_{j\in\omega}$.

Proposition 2.2.5 *If* $\alpha \Subset \beta$, *then Spec* $\alpha \subseteq$ *Spec* β.

Indeed, let us consider real hypernumbers $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ and $\beta = \mathrm{Hn}(b_i)_{i\in\omega}$ such that $(a_i)_{i\in\omega}$ is a subsequence of $(b_i)_{i\in\omega}$ and a real number $r \in$ *Spec* α. It means that $r = \lim_{i\to\infty} c_i$ for some subsequence of the sequence $(a_i)_{i\in\omega}$. Then the same is true for the sequence $(b_i)_{i\in\omega}$ because any subsequence of $(a_i)_{i\in\omega}$ is also a subsequence of $(b_i)_{i\in\omega}$. Consequently, any element from Spec α *also belongs to* Spec β.

Corollary 2.2.25 *Spec* $\alpha = \bigcup_{\gamma\in\mathrm{Sub}\,\alpha}$ *Spec* γ.

Proposition 2.2.6 *If* $\alpha \Subset \beta$ *and* α *is an oscillating finite real hypernumber, then* β *is an oscillating real hypernumber.*

Indeed, if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}, \beta = \mathrm{Hn}(b_i)_{i\in\omega}, (a_i)_{i\in\omega}$ is a subsequence of $(b_i)_{i\in\omega}$ and there is a number $k \in \boldsymbol{R}^{++}$ such that there are two infinite sequences of natural numbers $m(i)$ and $n(i)$ with $i = 1, 2, 3, \ldots$ such that $a_{m(i)} - a_{n(i)} > k$ and $a_{m(i+1)} - a_{n(i)} > k$ for all $i = 1, 2, 3, \ldots$, then the same is true for the elements b_i because $(a_i)_{i\in\omega}$ is a subsequence of $(b_i)_{i\in\omega}$.

Proposition 2.2.7 *If* $\alpha \Subset \beta$ *and* β *is an increasing (decreasing) real hypernumber, then* α *is an increasing (decreasing) real hypernumber.*

Indeed, if we have an increasing (decreasing) sequence of real numbers, then any of its subsequence is also increasing (decreasing).

Proposition 2.2.8 *If* β *is an increasing (decreasing) infinite real hypernumber, then for any hypernumber* $\alpha \in \boldsymbol{R}_\omega$*, there is a subhypernumber* γ *of hypernumber* β *such that* $\gamma \geq \alpha (\gamma \leq \alpha)$.

Proof is left as an exercise.

Proposition 2.2.9 *A hypernumber* α *is separated from* 0 *if and only if all of its subhypernumbers are separated from* 0.

Indeed, if α is separated from 0, then there is a sequence $(a_i)_{i\in\omega}$ of real numbers such that $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ and $|a_i| > k > 0$ for some number k and all $i \in \omega$. However, the latter condition is true for any subsequence of the sequence $(a_i)_{i\in\omega}$. So, any subhypernumber of α is separated from 0. At the same time, the condition of the proposition is sufficient because α is its own subhypernumber.

Lemma 2.2.8 *If* $\alpha \Subset \beta$ *and* $\gamma \Subset \beta$*, then* $\gamma \Subset \beta$*, i.e., relation* $\Subset$ *is transitive.*

Proof is left as an exercise.

As relation $\Subset$ is reflexive, we have the following result.

Proposition 2.2.10 *Relation* $\Subset$ *is a preorder.*

In a general case, relation $\Subset$ is not a partial order (cf. Appendix). To show that relations $\alpha \Subset \beta$ and $\beta \Subset \alpha$, do not imply the relation $\alpha = \beta$ in a general case, we consider the following example.

Example 2.2.1 Let us consider two real sequences $\boldsymbol{a} = (1,0, 1, 0, 1, 0, 1, \ldots, 1, 0, 1, 0, \ldots)$ and $\boldsymbol{b} = (1,0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, \ldots, 1, 0, 0, 0, 1, 0, 0, 0, \ldots)$. Then $\boldsymbol{a}$ is a subsequence of $\boldsymbol{b}$ where each pair 1,0 from $\boldsymbol{a}$ is mapped into the corresponding pair

1,0 from $\boldsymbol{b}$. Consequently, the hypernumber α defined by $\boldsymbol{a}$ is a subhypernumber of the hypernumber β defined by $\boldsymbol{b}$, i.e., $\alpha \Subset \beta$. At the same time, there is a mapping f of the sequence $\boldsymbol{b}$ into the sequence $\boldsymbol{a}$ such that numbers 0 from $\boldsymbol{b}$ are mapped into numbers 0 from $\boldsymbol{a}$ that have the same number in $\boldsymbol{b}$, that is, the first number 0 from $\boldsymbol{b}$ is mapped into the first number 0 from $\boldsymbol{a}$, the second number 0 from $\boldsymbol{b}$ is mapped into the second number 0 from $\boldsymbol{a}$ and so on. At the same time, numbers 1 from $\boldsymbol{b}$ are mapped into numbers 1 from $\boldsymbol{a}$ so that when a number 1 from $\boldsymbol{b}$ is mapped, then the next numbers 1 from $\boldsymbol{b}$ is mapped into the number 1 from $\boldsymbol{a}$ that goes after three zeroes and two ones after the previous image, i.e., the first number 1 from $\boldsymbol{b}$ is mapped into the first number 1 from $\boldsymbol{a}$, the second number 1 from $\boldsymbol{b}$ is mapped into the fourth number 1 from $\boldsymbol{a}$ and so on. Consequently, the hypernumber β defined by $\boldsymbol{b}$ is a subhypernumber of the hypernumber α defined by $\boldsymbol{a}$, i.e., $\beta \Subset \alpha$. However, $\alpha \neq \beta$.

We know that if $\alpha \leq \beta$ and $\gamma \geq 0$, then $\alpha + \gamma \leq \beta + \gamma$. A similar property is not true for the relation $\Subset$ as the following example demonstrates.

Example 2.2.2 Let us consider two real sequences $\boldsymbol{a} = (2, 4, 6, 8, 10, \ldots)$ and $\boldsymbol{b} = (1, 2, 3, \ldots)$. Then $\boldsymbol{a}$ is a subsequence of $\boldsymbol{b}$. Consequently, the hypernumber $\alpha = \text{Hn}\, \boldsymbol{a}$ is a subhypernumber of the hypernumber $\beta = \text{Hn}\, \boldsymbol{b}$, i.e., $\alpha \Subset \beta$. Taking $\gamma = \text{Hn}\, \boldsymbol{b}$, we see that $\gamma \geq 0$. At the same time, $\alpha + \gamma = \text{Hn}(3, 6, 9, 12, 15, \ldots)$, while $\beta + \gamma = \text{Hn}(2, 4, 6, 8, 10, \ldots)$. Then any subhypernumber of the hypernumber $\beta + \gamma$ is defined by a sequence of even numbers, while a representative of $\alpha + \gamma$ contains infinitely many odd numbers. So, by Lemma 2.2.2, the hypernumber $\alpha + \gamma$ cannot be a subhypernumber of the hypernumber $\beta + \gamma$.

Theorem 2.1.2 implies the following result.

Proposition 2.2.11 $\alpha = \text{Hn}(a_i)_{i\in\omega}$ *is an oscillating real hypernumber if and only if it has, at least, two stable real subhypernumbers or one infinite monotone real hypernumber and, at least, one stable real subhypernumber or one infinite increasing and one infinite decreasing real hypernumbers.*

It is also possible to define new operations for real hypernumbers.

Definition 2.2.9 A real hypernumber α is called a *disjunctive union* of real hypernumbers β and γ if there are sequences $\boldsymbol{a} = (a_i)_{i\in\omega}$, $\boldsymbol{b} = (b_i)_{i\in\omega}$, and $\boldsymbol{c} = (c_i)_{i\in\omega}$ such that $\alpha = \text{Hn}(a_i)_{i\in\omega}$, $\beta = \text{Hn}(b_i)_{i\in\omega}$, $\gamma = \text{Hn}(c_i)_{i\in\omega}$, $\boldsymbol{b} = (b_i)_{i\in\omega}$, and $\boldsymbol{c} = (c_i)_{i\in\omega}$ are subsequences of the sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ such that images of the injections $f_{ba} : \omega \to \omega$ and $f_{ca} : \omega \to \omega$ do not have common elements and each element from $(a_i)_{i\in\omega}$ belongs either to $(b_i)_{i\in\omega}$ or to $(c_i)_{i\in\omega}$.

The disjunctive union of real hypernumbers β and γ is denoted by $\beta \bigcup \gamma$.

Proposition 2.2.12 *If* $\gamma \Subset \alpha$, *and* $\delta \Subset \beta$, *then* $\gamma \bigcup \delta \Subset \alpha \bigcup \beta$.

Proof is left as an exercise.

Proposition 2.2.13 $\mathit{Spec}(\beta \bigcup \gamma) = \mathit{Spec}\, \beta \bigcup \mathit{Spec}\, \gamma$.

Indeed, any converging subsequence of the sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ has infinitely many elements either from $\boldsymbol{b} = (b_i)_{i\in\omega}$ or from $\boldsymbol{c} = (c_i)_{i\in\omega}$ or from both of these sequences. Thus, its limit belongs either to Spec β or to Spec γ or to both of these sets.

In contrast to disjunctive union, spectrum of hypernumbers is not always distributive with respect to addition, i.e., the equality $Spec\ \alpha + Spec\ \beta = Spec(\alpha + \beta)$ is not always true. It is even possible that $Spec\,\alpha \neq \varnothing$ and $Spec\,\beta \neq \varnothing$, but $Spec(\alpha + \beta) = \varnothing$. Indeed, let us consider the following example.

$$\alpha = \mathrm{Hn}(a_i)_{i\in\omega} \quad \text{and} \quad \beta = \mathrm{Hn}(b_i)_{i\in\omega}$$

where

$$a_n = \begin{cases} k & \text{if } n = 2k \\ 1 & \text{if } n = 2k - 1 \end{cases}$$

and

$$b_n = \begin{cases} k & \text{if } n = 2k - 1 \\ 1 & \text{if } n = 2k \end{cases}$$

Then $Spec\ \alpha = Spec\ \beta = \{1\}$, while $Spec(\alpha + \beta) = \varnothing$.

Proposition 2.2.14 $a(\beta \bigcup \gamma) = a\beta \bigcup a\gamma$ for any real number a.

Proof is left as an exercise.

Note that the set $\boldsymbol{C}_\omega$ of all complex hypernumbers has similar algebraic properties.

To conclude this section, we show that surreal numbers cannot contain hypernumbers that are not separated from zero. Indeed, let us take two hypernumbers $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ and $\beta = \mathrm{Hn}(b_i)_{i\in\omega}$ in which $a_{2i} = 1$, $a_{2i-1} = 0$, $b_{2i} = 0$, and $b_{2i-1} = 1$, for all $i = 1, 2, 3, \ldots$ The system **No** of all surreal numbers is a field (Gonshor 1986). It means that any element δ from **No** has the inverse, i.e., an element γ such that $\delta\gamma = 1$. Thus, if $\boldsymbol{R}_\omega$ is homomorphically included in **No**, then there is an element η such that $\beta\eta = 1$. Then $\alpha(\beta\eta) = \alpha \cdot 1 = \alpha$. At the same time, $(\alpha\beta)\eta = 0{\cdot}\eta = 0$ because by the definition of multiplication of hypernumbers (Definition 2.2.3), we have $\alpha\beta = 0$. Because multiplication in a field is associative (Kurosh 1963), we come to a contradictory equality $1 = \alpha(\beta\eta) = (\alpha\beta)\eta = 0$. This contradiction shows that surreal numbers do not include real hypernumbers.

2.3 Topological Properties of Hypernumbers

Taking the natural projection $p : \boldsymbol{R}^\omega \to \boldsymbol{R}_\omega$, we will define a topology in the space $\boldsymbol{R}_\omega$ by means of neighborhoods that are images of neighborhoods from $\boldsymbol{R}^\omega$. Let us consider a positive real number k and $\boldsymbol{a} = (a_i)_{i\in\omega} \in \boldsymbol{R}^\omega$.

Definition 2.3.1 A *spherical neighborhood* of $\boldsymbol{a}$ is a set of the form $\mathrm{O}_k\,\boldsymbol{a}$ for some $k > 0$, where

$$\mathrm{O}_k\boldsymbol{a} = \{c = (c_i)_{i\in\omega} \in \boldsymbol{R}^\omega; \exists r \in \boldsymbol{R}^{++}\exists n \in \omega\forall i > n(|a_i - c_i| < k - r)\}$$

Note that a spherical neighborhood of a bounded sequence contains only bounded sequences.

The system T of all spherical neighborhoods determines a topology $\boldsymbol{\tau}$ in $\boldsymbol{R}^\omega$. In addition, T induces the interval topology for real numbers in $\boldsymbol{R}$ as a subset of $\boldsymbol{R}^\omega$. Thus, $\boldsymbol{\tau}$ is a natural extension of the interval topology in $\boldsymbol{R}$.

Proposition 2.3.1 *$\boldsymbol{R}^\omega$ is a topological vector space with respect to $\boldsymbol{\tau}$.*

Proof We remind that a vector (linear) space over a topological field F, i.e., over $\boldsymbol{R}$, is topological if operations of addition and multiplication by elements from F (from $\boldsymbol{R}$) are continuous mappings (Bourbaki 1987). At the same time, a mapping $g : X \to Y$ from a topological space X to a topological space Y is continuous if for any point y from Y and any of its neighborhood $\mathrm{O}y$, if $g(x) = y$, then there is a neighborhood $\mathrm{O}x$ of the point a such that $g(\mathrm{O}x) \subseteq \mathrm{O}y$ (Alexandroff 1961).

Let us take a sequence $\boldsymbol{c} = (c_i)_{i\in\omega} \in \boldsymbol{R}^\omega$, its spherical neighborhood $\mathrm{O}_k\,\boldsymbol{c}$ and assume that $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$, then for $h < \frac{1}{2}k$, the neighborhood $\mathrm{O}_h\boldsymbol{a} \times \mathrm{O}_h\boldsymbol{b}$ of the pair $(\boldsymbol{a}, \boldsymbol{b})$ is mapped by addition into the neighborhood $\mathrm{O}_k\,\boldsymbol{c}$, i.e., $\mathrm{O}_h\boldsymbol{a} + \mathrm{O}_h\boldsymbol{b} \subseteq \mathrm{O}_k\boldsymbol{c}$.

In addition, if $\boldsymbol{c} = r\boldsymbol{a}$, then for $h < \min\{\frac{1}{2}k, \frac{1}{2}\}$, the neighborhood $\mathrm{O}_h r \times \mathrm{O}_h\boldsymbol{a}$ of the pair $(r, \boldsymbol{a})$ where $\mathrm{O}_h r = (r - h, r + h)$ is mapped by multiplication into the neighborhood $\mathrm{O}_k\boldsymbol{c}$, i.e., $\mathrm{O}_h r \cdot \mathrm{O}_h\boldsymbol{a} \subseteq \mathrm{O}_k\boldsymbol{c}$, because $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} < 1$ when $k \geq 1$ and $(\frac{1}{2}k)(\frac{1}{2}k) = \frac{1}{4}k^2 < k$ when $k < 1$.

Proposition is proved.

Let us investigate which of the topological properties described in Appendix are possessed by the spaces $\boldsymbol{R}^\omega$ and $\boldsymbol{R}_\omega$.

Theorem 2.3.1 *The topological space $\boldsymbol{R}^\omega$ does not satisfy even the axiom* $\mathbf{T}_0$.

To prove the theorem, it is sufficient to take elements $\mathbf{0} = (a_i = 0)_{i\in\omega}$ and $\boldsymbol{b} = (b_i = 1/i)_{i\in\omega}$. Any spherical neighborhood of one of them includes the second point.

Definitions 2.1.4 and 2.3.1 imply a more general result.

Lemma 2.3.1 *Sequences $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}^\omega$ determine the same hypernumber if and only if every spherical neighborhood of $\boldsymbol{a}$ contains $\boldsymbol{b}$.*

Proof is left as an exercise.

As $\boldsymbol{R}_\omega$ is the quotient space of $\boldsymbol{R}^\omega$, the topology $\boldsymbol{\tau}$ induces on $\boldsymbol{R}_\omega$ the topology $\boldsymbol{\delta}$, which is generated by means of the projections of the spherical neighborhoods. Note that a spherical neighborhood of a finite hypernumber contains only finite hypernumbers.

Proposition 2.3.2 *The topology $\boldsymbol{\delta}$ satisfies the axiom* $\mathbf{T}_2$, *and thus, $\boldsymbol{R}_\omega$ is a Hausdorff space.*

Proof Let us consider two arbitrary points (hypernumbers) α and β from $\boldsymbol{R}_\omega$. If $\alpha = \beta$ in $\boldsymbol{R}_\omega$, then any sequences $\boldsymbol{a} = (a_i)_{i\in\omega} \in \alpha$ and $\boldsymbol{b} = (b_i)_{i\in\omega} \in \beta$ satisfy the following condition: there is a positive number k such that for any $n \in \omega$, there is an $i > n$ such that $|a_i - b_i| > k$. This condition makes it possible to choose an infinite set M of natural numbers such that for any $m \in M$ the inequality $|a_m - b_m| > k$ is valid.

Let us take $h = k/4$ and consider two spherical neighborhoods $\mathrm{O}_h\, \boldsymbol{a}$ and $\mathrm{O}_h\, \boldsymbol{b}$ of the points $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\boldsymbol{R}^\omega$. If $p : \boldsymbol{R}^\omega \to \boldsymbol{R}_\omega$ is the natural projection, i.e., for any sequences $\boldsymbol{a} = (a_i)_{i\in\omega}$, $p(\boldsymbol{a}) = \mathrm{Hn}(a_i)_{i\in\omega}$, then the projections $p(\mathrm{O}_h\boldsymbol{a})$ and $p(\mathrm{O}_h\boldsymbol{b})$ of these neighborhoods are neighborhoods $\mathrm{O}_h\alpha$ and $\mathrm{O}_h\beta$ of α and β with respect to the topology $\boldsymbol{\delta}$. By construction, $p(\mathrm{O}_h a) \cap p(\mathrm{O}_h b) = \varnothing$. To prove this, we suppose that this is not true. Then there is a hypernumber $\gamma \in \boldsymbol{R}_\omega$ that is an element of the set $p(\mathrm{O}_h\boldsymbol{a}) \cap p(\mathrm{O}_h\boldsymbol{b})$. It implies that there are sequences $\boldsymbol{u} = (u_i)_{i\in\omega}$ and $\boldsymbol{v} = (v_i)_{i\in\omega}$ from $\boldsymbol{R}^\omega$ for which $p(\boldsymbol{u}) = p(\boldsymbol{v}) = \gamma$, $\boldsymbol{u} \in \mathrm{O}_h\boldsymbol{a}$ and $\boldsymbol{v} \in \mathrm{O}_h b$.

The equality $p(\boldsymbol{u}) = p(\boldsymbol{v})$ implies that for the chosen number k, the following condition is valid: $\exists\, m \in \omega\ \forall i > m(|u_i - v_i| < k/3)$. The set M, which is determined above, is infinite. So, there is $j \in M$ such that it is greater than m and $|u_j - v_j| \geq |a_j - b_j| - |a_j - u_j| - |b_j - v_j| - k/4 - k/4 = k/2 > k/3$ because $a_j - b_j = a_j - u_j + u_j - v_j + v_j - b_j$, $|v_j - b_j| = |b_j - v_j|$and $|a_j - b_j| \leq |a_j - u_j| + |u_j - v_j| + |b_j - v_j|$ because.

It contradicts the condition $|u_j - v_j| < k/3$.

Consequently, the assumption is not true, and $p(\mathrm{O}_h\boldsymbol{a}) \cap p(\mathrm{O}_h\boldsymbol{b}) = \varnothing$. Proposition is proved because α and β are arbitrary points from the space $\boldsymbol{R}_\omega$.

Proposition is proved.

This result makes it possible to obtain a characteristic property for $\boldsymbol{R}_\omega$.

Theorem 2.3.2 *$\boldsymbol{R}_\omega$ is the largest Hausdorff quotient space of the topological space $\boldsymbol{R}^\omega$.*

Proof By Proposition 3.2, $\boldsymbol{R}_\omega$ is a Hausdorff space. Thus, to prove the theorem, it is necessary to demonstrate that if a Hausdorff space X is a quotient space of $\boldsymbol{R}^\omega$ with the projection $q : \boldsymbol{R}^\omega \to X$, then there is a continuous projection $v : \boldsymbol{R}_\omega \to X$ for which $q = pv$, i.e., the following diagram is commutative:

$$\begin{array}{ccc} \boldsymbol{R}^\omega & \xrightarrow{p} & \boldsymbol{R}_\omega \\ q \downarrow & \swarrow \nu & \\ X & & \end{array}$$

Let us consider a Hausdorff space X with the continuous projection $q : \boldsymbol{R}^\omega \to X$. Then, for any points $x, y \in X$, the inequality $x \neq y$ implies existence of neighborhoods $\mathrm{O}x$ and $\mathrm{O}y$ for which $\mathrm{O}x \cap \mathrm{O}y = \varnothing$. As X is a quotient space of $\boldsymbol{R}^\omega$, there are points $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}^\omega$ for which $q(\boldsymbol{a}) = x$ and $q(\boldsymbol{b}) = y$. The inverse images

$q^{-1}(\mathrm{O}x)$ and $q^{-1}(\mathrm{O}y)$ are open sets because q is a continuous mapping (Kelly 1957). Besides, $\boldsymbol{a} \in q^{-1}(\mathrm{O}x)$ and $\boldsymbol{b} \in q^{-1}(\mathrm{O}y)$ because $q(\boldsymbol{a}) = x$ and $q(\boldsymbol{b}) = y$. That is why, for some k, the set $q^{-1}(\mathrm{O}x)$ contains the spherical neighborhood $\mathrm{O}_k\,\boldsymbol{a}$ of $\boldsymbol{a}$ and $q^{-1}(\mathrm{O}y)$ contains the spherical neighborhood $\mathrm{O}_k\boldsymbol{b}$ of $\boldsymbol{b}$.

Let us suppose that $p(\boldsymbol{a}) = p(\boldsymbol{b})$. Then $\boldsymbol{a} \in \mathrm{O}_k\boldsymbol{b}$ and $\boldsymbol{b} \in \mathrm{O}_k\boldsymbol{a}$. As a consequence, $x = q(\boldsymbol{a}) \in q(\mathrm{O}_k\boldsymbol{b}) \subseteq \mathrm{O}y$ and $y = q(\boldsymbol{b}) \in q(\mathrm{O}_k\boldsymbol{a}) \subseteq \mathrm{O}x$. It contradicts to the condition that $\mathrm{O}x \cap \mathrm{O}y = \varnothing$. Thus, for arbitrary x and y from X and such $\boldsymbol{a}$, $\boldsymbol{b}$ from $\boldsymbol{R}^{\omega}$ that $q(\boldsymbol{a}) = x$ and $q(\boldsymbol{b}) = y$, we have $p(\boldsymbol{a}) \neq p(\boldsymbol{b})$.

It makes it possible to define the mapping $v : \boldsymbol{R}_{\omega} \to X$ as follows: $v(x) = p(\boldsymbol{a})$ for any $x \in X$ and for a point $\boldsymbol{a} \in \boldsymbol{R}^{\omega}$ such that $q(\boldsymbol{a}) = x$. The definition of v implies that the mapping v is continuous because the topology in $\boldsymbol{R}_{\omega}$ is induced by the topology in $\boldsymbol{R}^{\omega}$ and $pv = q$.

Theorem is proved.

Theorem 2.3.2 makes it possible to obtain an axiomatic characterization of real hypernumbers.

Theorem 2.3.3 *$\boldsymbol{R}_{\omega}$ is the biggest extension of $\boldsymbol{R}$ that has the following properties:*

(THN1) Any sequence of real numbers is associated with some element from $\boldsymbol{R}_{\omega}$.
(THN2) The topology in $\boldsymbol{R}_{\omega}$ *is induced by a natural extension of the topology in* $\boldsymbol{R}$ *to the topology in* $\boldsymbol{R}^{\omega}$.
(THN3) The topology in $\boldsymbol{R}_{\omega}$ *is Hausdorff.*

This result demonstrates that in contrast to hyperreal numbers from nonstandard analysis, the construction of hypernumbers is completely based on topological principles.

Proposition 2.3.1 and properties of the operations obtained in Sect. 2.2 imply the following result.

Theorem 2.3.4 *$\boldsymbol{R}_{\omega}$ is a topological vector space with respect to* $\boldsymbol{\delta}$.

In addition to topology, it is possible to define more deep-seated structures in $\boldsymbol{R}_{\omega}$. *Let us take* $\boldsymbol{R}_{\omega}^{+} = \{\alpha \in \boldsymbol{R}_{\omega}; \alpha \geq 0\}$ *and assume that* X *is a vector space over* $\boldsymbol{R}$.

Definition 2.3.2

(a) A mapping $q : X \to \boldsymbol{R}_{\omega}^{+}$ is called a *hypernorm* if it satisfies the following conditions:

N1 $q(x) = 0$ if and only if $x = 0$.
N2 $q(ax) = |a| \cdot q(x)$ for any x from X and any number a from $\boldsymbol{R}$.
N3 (the triangle inequality)

$$q(x + y)(x) + q(y) \quad \text{for any } x \text{ and } y \text{ from } X$$

(b) A vector space L with a norm is called a *hypernormed vector space* or simply, a *hypernormed space*.
(c) The real hypernumber $q(x)$ is called the *hypernorm* of an element x from the hypernormed space X.

Note that any norm is a hypernorm that takes values only in the set of real numbers.

Theorem 2.3.5 $\boldsymbol{R}_\omega$ *is a hypernormed space.*

Proof For a real hypernumber $\alpha = \text{Hn}(a_i)_{i\in\omega}$, we define its hypernorm as $||\alpha|| = \text{Hn}(|a_i|)_{i\in\omega}$ where $|a|$ is an absolute value of the real number a. Then the hypernorm of a real hypernumber does not depend on the choice of its representative. Indeed, if $\alpha = \text{Hn}(a_i)_{i\in\omega} = \text{Hn}(d_i)_{i\in\omega}$, then

$$\lim_{i\to\infty}|a_i - d_i| = 0$$

By properties of absolute value, we have $||a_i| - |d_i|| \leq |a_i - d_i|$. Thus, $\lim_{i\to\infty}||a_i| - d_i|| = 0$ and $\text{Hn}(|a_i|)_{i\in\omega} = \text{Hn}(|d_i|)_{i\in\omega}$.

Consequently, if $\alpha = 0$, then $||a|| = 0$. If $||a|| = 0$, then $\lim_{i\to\infty}|a_i| = 0$ and thus, $\alpha = 0$. Condition N1 is proved. Condition N2 directly follows from Theorem 2.2.1. Besides, if $\alpha = \text{Hn}(a_i)_{i\in\omega}$ and $\beta = \text{Hn}(b_i)_{i\in\omega}$, then $||\alpha + \beta|| = \text{Hn}(|a_i + b_i|)_{i\in\omega} \leq \text{Hn}(|a_i| + |b_i|)_{i\in\omega} = \text{Hn}(|a_i|)_{i\in\omega} + \text{Hn}(|b_i|)_{i\in\omega} = ||\alpha|| + ||\beta||$ because $|a_i + b_i| \leq |a| + |b|$ for any real numbers a and b. This gives us condition N3.

Theorem is proved.

Let X be some set.

Definition 2.3.3

(a) A mapping $\mathbf{d} : X \times X \to \boldsymbol{R}_\omega^+$ is called a *hypermetric* (or a *hyperdistance function*) if it satisfies the following axioms:

M1 $\mathbf{d}(x, y) = 0$ if and only if $x = y$.
M2 $\mathbf{d}(x, y) = \mathbf{d}(y, x)$ for all $x, y \in X$.
M3 $\mathbf{d}(x, y) \leq \mathbf{d}(x, z) + \mathbf{d}(z, y)$ for all $x, y, z \in X$.

(b) A set X with a hypermetric $\mathbf{d}$ is called a *hypermetric space*.

(c) The real hypernumber $\mathbf{d}(x, y)$ is called the *distance* between x and y in the hypermetric space X.

Note that the distance between two elements in a hypermetric space can be a real number, finite hypernumber or infinite hypernumber.

Lemma 2.3.2 *A hypernorm in a vector space L induces a hypermetric in this space.*

Indeed, if $q : X \to \boldsymbol{R}_\omega^+$ is a hypernorm in L and x and y are elements from L, then we can define $\mathbf{d}(x, y) = q(x - y)$. Properties of a norm imply that $\mathbf{d}$ satisfies all axioms M1–M3.

Theorem 2.3.5 and Lemma 2.3.2 imply the following result.

Corollary 2.3.1 $\boldsymbol{R}_\omega$ *is a hypermetric space.*

It is necessary to remark that in this book, we consider only the simplest case of hypernumbers. More general classes of hypernumbers are introduced and explored in Burgin (2001, 2005c).

Chapter 3
Extrafunctions

It is the function of art to renew our perception. What we are familiar with we cease to see. The writer shakes up the familiar scene, and, as if by magic, we see a new meaning in it.

(Anais Nin 1903–1977).

In this chapter, we define and study different types of extrafunctions. It would be natural to speak of hyperfunctions instead of extrafunctions as mappings of hypernumbers. However, the term *hyperfunction* is already used in mathematics. So, we call mappings of hypernumbers by the name *extrafunction*. The main emphasis here is on general extrafunctions and norm-based extrafunctions, which include conventional distributions, hyperdistributions, restricted pointwise extrafunctions, and compactwise extrafunctions, which have studied before in different publications. In Sect. 3.1, the main constructions are described and their basic properties are explicated. For instance, a criterion is found (Theorem 3.1.1) for existence of an extension of the conventional functions to general extrafunctions. Relations between norm-based extrafunctions, distributions, hyperdistributions, pointwise extrafunctions, and compactwise extrafunctions are established. In Sect. 3.2, various algebraic properties of norm-based extrafunctions are obtained. For instance, it is demonstrated (Theorem 3.2.1) that norm-based extrafunctions form a linear space over the field of real numbers $\boldsymbol{R}$. It is also proved (Theorem 3.2.5) that the class of all bounded norm-based extrafunctions is a linear algebra over $\boldsymbol{R}$. In Sect. 3.3, we study topological properties of norm-based extrafunctions. For instance, it is demonstrated (Theorem 3.3.1) that norm-based extrafunctions form a Hausdorff topological space.

3.1 Definitions and Typology

There are several types of extrafunctions. The most straightforward approach gives us real pointwise extrafunctions also called general extrafunctions. Topological constructions lead to various classes of norm-based extrafunctions, which include

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distributions, extended distributions, hyperdistributions, restricted pointwise extrafunctions, and compactwise extrafunctions. In what follows, we consider, as a rule, only total real functions, i.e., functions defined everywhere in $\boldsymbol{R}$.

3.1.1 General Extrafunctions

Taking the definition of real functions and changing the set of real numbers $\boldsymbol{R}$ to the set of real hypernumbers $\boldsymbol{R}_\omega$, we come to general extrafunctions.

Definition 3.1.1 A partial mapping $f : \boldsymbol{R}_\omega \to \boldsymbol{R}_\omega$ is called a *real pointwise extrafunction* or a *real general extrafunction*.

We denote by $F(\boldsymbol{R}_\omega, \boldsymbol{R}_\omega)$ the set of all (general) real pointwise extrafunctions.

As we know, functions are usually defined by formulas, e.g., $f(x) = x^5$, $f(x) = \mathrm{e}^x$, or $f(x) = 7 \ln x + 3 \sin x$. It is natural to ask a question what formulas can be extended to general extrafunctions. For instance, is it possible to define the function e^x for hypernumbers so that its restriction on the set of real numbers is the conventional exponential function e^x? The following definition explains what a correct extension of a function from numbers to hypernumbers is and the result of Theorem 3.1.1 gives an answer to the above question.

Definition 3.1.2

(a) A real function f is *correctly defined* for a real hypernumber α represented in a set $X \subseteq \boldsymbol{R}$ if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega} = \mathrm{Hn}(b_i)_{i\in\omega}$ implies $\mathrm{Hn}(f(a_i))_{i\in\omega} = \mathrm{Hn}(f(b_i))_{i\in\omega} = f(\alpha)$.

(b) A real function f is *correctly defined* in X if f is correctly defined for any hypernumber represented in X.

(c) If a real function $f(x)$ is correctly defined for a real hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$, then the *extrafunction extension* $F(x)$ of the function $f(x)$ is defined by the formula

$$F(\alpha) = \mathrm{Hn}(f(a_i))_{i\in\omega}$$

Example 3.1.1 The function $f(x) = x$ is correctly defined in $\boldsymbol{R}$. Indeed, if $\mathrm{Hn}(a_i)_{i\in\omega} = \mathrm{Hn}(b_i)_{i\in\omega}$, then $\mathrm{Hn}(f(a_i))_{i\in\omega} = \mathrm{Hn}(a_i)_{i\in\omega} = \mathrm{Hn}(b_i)_{i\in\omega} = \mathrm{Hn}(f(b_i))_{i\in\omega}$. It means that we can correctly define the function $f(x) = x$ for any hypernumber, i.e., to correctly extend it to the whole $\boldsymbol{R}_\omega$.

Example 3.1.2 The function $f(x) = \mathrm{e}^x$ is correctly defined in the interval $[a, b]$ for any positive real numbers a and b. Indeed, if $\mathrm{Hn}(a_i)_{i\in\omega} = \mathrm{Hn}(b_i)_{i\in\omega}$ and $a_i, b_i \in [a, b]$ for all $i = 1, 2, 3, \ldots$, then

$$\begin{aligned} |\mathrm{e}^{a_i} - \mathrm{e}^{b_i}| &= |\mathrm{e}^{a_i} \cdot (1 - \mathrm{e}^{b_i - a_i})| = |\mathrm{e}^{a_i}| \cdot |1 - \mathrm{e}^{b_i - a_i}| = \mathrm{e}^{a_i} \cdot |1 - \mathrm{e}^{b_i - a_i}| \\ &\leq \mathrm{e}^{b} \cdot |1 - \mathrm{e}^{b_i - a_i}| \end{aligned}$$

as e^x is a monotonous function in $\boldsymbol{R}$. The equality $\mathrm{Hn}(a_i)_{i\in\omega} = \mathrm{Hn}(b_i)_{i\in\omega}$ implies $\lim_{i\to\infty}|a_i - b_i| = 0$.

Consequently, $\lim_{i\to\infty} e^{b_i - a_i} = 1$ and $\lim_{i\to\infty}|1 - e^{b_i - a_i}| = 0$. Thus,

$$0 \leq \lim_{i\to\infty}|e^{a_i} - e^{b_i}| \leq \lim_{i\to\infty} e^b \cdot |1 - e^{b_i - a_i}| = e^b \cdot \lim_{i\to\infty}|1 - e^{b_i - a_i}| = 0$$

By Definition 2.3.1, we have $\mathrm{Hn}(e^{ai})_{i\in\omega} = \mathrm{Hn}(e^{bi})_{i\in\omega}$. It means that we can correctly define the function e^x for any bounded hypernumber.

Example 3.1.3 Let us consider the function $f(x) = [x]$ where $[x]$ is the largest integer less than x. Then $\mathrm{Hn}(1 - (1/i))_{i\in\omega} = \mathrm{Hn}(b_i)_{i\in\omega} = 1$ where all $b_i = 1$. However, $[1 - (1/i)] = 0$ for all $i = 1, 2, 3\ldots$ and thus,

$$\begin{aligned}\mathrm{Hn}(f(1 - (1/i))_{i\in\omega} &= \mathrm{Hn}([1 - (1/i)])_{i\in\omega} = 0 \neq \mathrm{Hn}(f(b_i))_{i\in\omega} = \mathrm{Hn}([b_i])_{i\in\omega} \\ &= \mathrm{Hn}(b_i)_{i\in\omega} = 1\end{aligned}$$

i.e., as an extrafunction $f(x) = [x]$ is not correctly defined in any interval $[a, b]$ if $[0, 1] \subseteq [a, b]$.

Theorem 3.1.1 *A real function f is correctly defined for every real hypernumber α represented in a closed (finite or infinite) interval $I \subseteq \boldsymbol{R}$ if and only if f is uniformly continuous in the interval I and is continuous at the point a when $I = [a, \infty)$, at the point b when $I = (-\infty, b]$, and at the points a and b when $I = [a, b]$.*

Proof
Sufficiency

1. Let us consider a uniformly continuous on the interval I real function $f : \boldsymbol{R} \to \boldsymbol{R}$ and a real hypernumber α represented in the interval I when $I = (-\infty, \infty)$. Then by definition, α is an arbitrary real hypernumber.
   Suppose

$$\alpha = \mathrm{Hn}(a_i)_{i\in\omega} = \mathrm{Hn}(b_i)_{i\in\omega}$$

Then

$$\lim_{i\to\infty}|a_i - b_i| = 0$$

We define

$$f(\alpha) = \mathrm{Hn}(f(a_i))_{i\in\omega}$$

and compare sequences $\mathrm{Hn}(f(a_i))_{i\in\omega}$ and $\mathrm{Hn}(f(b_i))_{i\in\omega}$

As f is a uniformly continuous real function in $\boldsymbol{R}$, for a given $\varepsilon > 0$, there is $\delta > 0$, such that $|a_i - b_i| < \delta$ implies $|f(a_i) - f(b_i)| < \varepsilon$. As $\lim_{i\to\infty}|a_i - b_i| = 0$, there is a natural number n such that for any $i > n$, we have

$$|a_i - b_i| < \delta$$

Thus,

$$|f(a_i) - f(b_i)| < \varepsilon$$

for all $i > n$. Then by definition,

$$\mathrm{Hn}(f(a_i))_{i \in \omega} = \mathrm{Hn}(f(b_i))_{i \in \omega}$$

It means that the function f is correctly defined for any real hypernumber α.

2. Now let us take $I = [a, \infty)$, a real function f that is uniformly continuous in the interval I and continuous at the point a, and a real hypernumber α represented in the interval I. Then there is a sequence $\boldsymbol{a} = (a_i)_{i \in \omega}$ of real numbers such that $a_i \in I$ for all $i \in \omega$ and $\alpha = \mathrm{Hn}(a_i)_{i \in \omega}$. In this situation, we have two cases: (1) the hypernumber α is separated from a; (2) the hypernumber α is not separated from a.

   In the first case, there is a sequence $(c_i)_{i \in \omega}$ of real numbers such that $\alpha = \mathrm{Hn}(c_i)_{i \in \omega}$ and $|a - c_i| > k > 0$ for some number k and all $i \in \omega$. If almost all elements c_i are less than a, then we have $|a_i - c_i| > |a - c_i| > k$ because all a_i belong to the interval I and thus, are larger than a. This is impossible because the equality $\alpha = \mathrm{Hn}(a_i)_{i \in \omega} = \mathrm{Hn}(c_i)_{i \in \omega}$ implies $\lim_{i \to \infty} |a_i - c_i| = 0$. Thus, we can assume that $a_i - a > k > 0$ for some number k and all $i \in \omega$.

   Suppose

$$\alpha = \mathrm{Hn}(a_i)_{i \in \omega} = \mathrm{Hn}(b_i)_{i \in \omega}$$

   where $a_i \in I$ for all $i \in \omega$.

   Then

$$\lim_{i \to \infty} |a_i - b_i| = 0$$

   and as $a_{\mathrm{i}} - a > k > 0$ for all $i \in \omega$, it is possible to assume that $b_i \in I$ for all $i \in \omega$. We define

$$f(\alpha) = \mathrm{Hn}(f(a_i))_{i \in \omega}$$

   and compare sequences $\mathrm{Hn}(f(a_i))_{i \in \omega}$ and $\mathrm{Hn}(f(b_i))_{i \in \omega}$.

   As f is a uniformly continuous real function in the interval I, for a given $\varepsilon > 0$, there is $\delta > 0$, such that $|a_i - b_i| < \delta$ implies $|f(a_i) - f(b_i)| < \varepsilon$. As $\lim_{i \to \infty} |a_i - b_i| = 0$, there is a natural number n such that for any $i > n$, we have

$$|a_i - b_i| < \delta$$

   Thus,

$$|f(a_i) - f(b_i)| < \varepsilon$$

for all $i > n$. Then by definition,

$$\mathrm{Hn}(f(a_i))_{i\in\omega} = \mathrm{Hn}(f(b_i))_{i\in\omega}$$

It means that the function f is correctly defined for the real hypernumber α.
In the case (2), there is a subsequence $(d_i)_{i\in\omega}$ of the sequence $(a_i)_{i\in\omega}$ such that

$$a = \lim_{i\to\infty} d_i$$

If almost all b_i are larger than a, then the proof that $\mathrm{Hn}(f(a_i))_{i\in\omega} = \mathrm{Hn}(f(b_i))_{i\in\omega}$ is the same as in the case (1) because f is a uniformly continuous real function in the interval I. However, this is not always true because $a = \lim_{i\to\infty} d_i$.

As $\lim_{i\to\infty}|a_i - b_i| = 0$, there is a subsequence $(c_i)_{i\in\omega}$ of the sequence $(b_i)_{i\in\omega}$ such that $a = \lim_{i\to\infty} c_i$ and almost all c_i are less than a.

As a result, we have two cases: (a) $a = \lim_{i\to\infty} a_i$; and (b) the sequence $(a_i)_{i\in\omega}$ is the union of two disjoint subsequences $(d_i)_{i\in\omega}$ and $(u_i)_{i\in\omega}$ where all $u_i > a + k$ for some natural number $k > 0$. In a similar way, when the case (a) is valid, $a = \lim_{i\to\infty} a_i$, while when the case (b) is valid, the sequence $(b_i)_{i\in\omega}$ is the union of two disjoint subsequences $(c_i)_{i\in\omega}$ and $(v_i)_{i\in\omega}$ where all $v_i > a + l$ for some natural number $l > 0$.

At first, let us consider the case (a). By the definition of hypernumbers, we have

$$\alpha = \mathrm{Hn}(a_i)_{i\in\omega} = \mathrm{Hn}(b_i)_{i\in\omega} = a$$

Then

$$\mathrm{Hn}(f(a_i))_{i\in\omega} = \mathrm{Hn}(f(b_i))_{i\in\omega} = f(a)$$

because the function f is continuous at the point a.

The case (b) is more sophisticated. As f is a uniformly continuous real function in the interval I, for a given $\varepsilon > 0$, there is $\delta > 0$, such that $|u_i - v_i| < \delta$ implies $|f(u)_i - f(v_i)| < \varepsilon$. As $\lim_{i\to\infty}|a_i - b_i| = 0$ implies $\lim_{i\to\infty}|u_i - v_i| = 0$, there is a natural number n such that for any $i > n$, we have

$$|u_i - v_i| < \delta$$

Thus,

$$|f(u_i) - f(v_i)| < \varepsilon$$

for all $i > n$. At the same time, as the function f that is continuous at the point a, for the same $\varepsilon > 0$, there is $\gamma > 0$, such that $|x - a| < \gamma/2$ implies $|f(x) - f(a)| < \varepsilon/2$. Thus, if

$$|d_i - c_i| \leq |d_i - a| + |a - c_i| < \gamma/2 + \gamma/2 = \gamma$$

then we have

$$|f(d_i) - f(c_i)| \leq |f(d_i) - f(a)| + |f(a) - f(c_i)| < \gamma/2 + \gamma/2 = \gamma$$

As $\lim_{i\to\infty} d_i = \lim_{i\to\infty} c_i = a$, there is a natural number m such that for any $i > m$, we have

$$|d_i - c_i| < \gamma$$

Consequently,

$$|f(d_i) - f(c_i)| < \varepsilon$$

for all $i > m$. As a result, there is a natural number p such that for any $i > p$, we have

$$|f(a_i) - f(b_i)| < \varepsilon$$

Then by definition,

$$\mathrm{Hn}(f(a_i))_{i\in\omega} = \mathrm{Hn}(f(b_i))_{i\in\omega}$$

It means that the function f is correctly defined for any real hypernumber α represented in the interval I.

3. Situations when $I = (-\infty, b]$ and $I = [a, b]$ are treated in a similar way.
Sufficiency is proved.

Necessity

1. Let us assume that f is not uniformly continuous on the interval I. In this case, there is $\varepsilon > 0$, such that there are two sequences $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ with all a_i and b_i from the interval I, $|a_i - b_i| \to 0$ when $i \to \infty$ but $|f(a_i) - f(b_i)| > \varepsilon$ for all i. Then

$$\mathrm{Hn}(a_i)_{i\in\omega} = \mathrm{Hn}(b_i)_{i\in\omega}$$

but

$$\mathrm{Hn}(f(a_i))_{i\in\omega} \neq \mathrm{Hn}(f(b_i))_{i\in\omega}$$

i.e., the real function f is not correctly defined for the real hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ represented in the interval I.

2. Now let us suppose that $I = [a, \infty)$ and f is not continuous at the point a. In this case, there is a sequence $(b_i)_{i\in\omega}$ such that $a = \lim_{i\to\infty} b_i$, but

$f(a) \neq \lim_{i\to\infty} f(b_i)$. Taking the sequence $(a_i)_{i\in\omega}$ such that $a_i = a$ for all $i \in \omega$, we have

$$\mathrm{Hn}(a_i)_{i\in\omega} = \mathrm{Hn}(b_i)_{i\in\omega} = a$$

but

$$f(a) = \mathrm{Hn}(f(a_i))_{i\in\omega} \neq \mathrm{Hn}(f(b_i))_{i\in\omega}$$

i.e., the real function f is not correctly defined for the real hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ represented in the interval I.

3. Situations when $I = (-\infty, b]$ and $I = [a, b]$ are treated in a similar way.

Necessity and thus the whole Theorem 3.1.1 are proved.

The extension of a real function $f : \boldsymbol{R} \to \boldsymbol{R}$ is denoted by $F : \boldsymbol{R}_\omega \to \boldsymbol{R}_\omega$.

Corollary 3.1.1 *If a real function f is continuous in a closed interval* $I = [a, b]$ *and at its ends* $a, b \in \boldsymbol{R}$, *then f is correctly defined for any real hypernumber* α *represented in this interval I and thus, it can be correctly defined in this interval.*

Indeed, any real function f continuous in a closed interval is uniformly continuous in this interval and the statement directly follows from Theorem 3.1.1.

Corollary 3.1.2 *If a real function f is continuous in* $\boldsymbol{R}$, *then f is correctly defined for any bounded real hypernumber* α *and thus, it can be correctly defined in any finite interval.*

Indeed, any bounded real hypernumber α is represented in a closed finite interval $I = [a, b]$ where $a, b \in \boldsymbol{R}$ and the function f is continuous in I. Thus, the statement directly follows from Corollary 3.1.1.

Such functions as polynomials, exponents e^{kx}, the modulus $|x|$, sin x and cos x can be correctly extended/defined in any interval $[a, b]$, while logarithms $\log_c x$ and the radical $\sqrt{x}$ can be correctly extended/defined in any interval $[a, b]$ with $a > 0$. For instance, if $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$, then $|\alpha| = \mathrm{Hn}(|a_i|)_{i\in\omega}$.

Correctly defined extrafunction extensions preserve many properties of the corresponding real functions.

Definition 3.1.3 A general real extrafunction $G : \boldsymbol{R}_\omega \to \boldsymbol{R}_\omega$ is called *monotone* (*antitone*) if $\alpha \leq \beta$ implies $G(\alpha) \leq G(\beta)(G(\beta) \leq G(\alpha))$ for any real hypernumbers α and β.

Proposition 3.1.1 *If a continuous real function f is monotone (antitone), then its extrafunction extension F also is monotone (antitone).*

Proof Let us take a continuous monotone real function f and two real hypernumbers $\alpha = (a_i)_{i\in\omega}$ and $\beta = (b_i)_{i\in\omega}$ such that $\alpha \leq \beta$. Then by Theorem 3.1.1, $F(\alpha) = \mathrm{Hn}(f(a_i))_{i\in\omega}$ and $F(\beta) = \mathrm{Hn}(f(b_i))_{i\in\omega}$. As $\alpha \leq \beta$, we have $a_i \leq b_i$

for almost all $i \in \omega$. As the function *f* is monotone, we have $f(a_i) \le f(b_i)$ for almost all $i \in \omega$. It means that $F(\alpha) \le F(\beta)$, i.e., F is a monotone extrafunction.

Proposition is proved.

Proposition 3.1.2 *If f is a non-negative (non-positive) continuous real function, then its extrafunction extension F is a non-negative (non-positive) real function.*

Indeed, if $\alpha = \mathrm{Hn}(f(a_i))_{i\in\omega}$, then $F(\alpha) = \mathrm{Hn}(f(a_i))_{i\in\omega}$. If $f(a_i) \ge 0$ for all $i = 1, 2, 3, \ldots$, then by Definition 2.2.2, $F(\alpha) \ge 0$.

Proof for non-positive real functions is similar.

Corollary 3.1.3 *If f is a bounded continuous real function, then its extrafunction extension F is also bounded.*

Proof is left as an exercise.

Proposition 3.1.3 *If f is an even (odd) continuous real function, then its extrafunction extension F is an even (odd) real extrafunction.*

Indeed, if *f* is an even real function, then

$$F(-\alpha) = (f(-a_i)_{i\in\omega}) = (f(a_i)_{i\in\omega}) = F(\alpha)$$

i.e., F is an even real extrafunction.

Proof for odd functions is similar.

Definition 3.1.4 A general real extrafunction $G : \boldsymbol{R}_\omega \to \boldsymbol{R}_\omega$ has a *period* γ if $G(\alpha + \gamma) = G(\alpha)$ for any real hypernumber α.

Proposition 3.1.4 *If f is a continuous periodic real function, then its extrafunction extension F is also a periodic real extrafunction.*

Proof Let us take a continuous periodic real function *f* with a period *k* and a real hypernumber $\alpha = (a_i)_{i\in\omega}$. Then by Theorem 3.1.1, $F(\alpha) = \mathrm{Hn}(f(a_i))_{i\in\omega}$. As the function *f* has a period *k*, we have $f(a_i + k) = f(a_i)$ for all $i \in \omega$. It means that $F(\alpha + k) \le F(\alpha)$, i.e., F is periodic real extrafunction with a period *k*.

Proposition is proved.

By Theorem 3.1.1, there is a correctly defined extrafunction extension of the absolute value $|x|$ of real numbers to the absolute value $|\alpha|$ of real hypernumbers, namely, $|\mathrm{Hn}(a_i)_{i\in\omega}| = \mathrm{Hn}(|a_i|)_{i\in\omega}$. This allows us to define Lipschitz continuity and the contraction property for extrafunctions.

Proposition 3.1.5 *If a real function f is Lipschitz continuous for a number* K*, then its extrafunction extension F is also Lipschitz continuous for the same number* K.

Indeed, if $\alpha = (a_i)$, $\beta = (b_i)$, and $|f(x) - f(y)| \le K|x - y|$ for all real numbers *x* and *y*, then by Theorem 3.1.1, Proposition 2.2.1, and Corollary 2.2.1, we have

$$\begin{aligned} |F(\alpha) - F(\beta)| &= |\mathrm{Hn}(f(a_i))_{i\in\omega} - \mathrm{Hn}(f(b_i))_{i\in\omega}| = |\mathrm{Hn}(f(a_i) - f(b_i))_{i\in\omega}| \\ &= \mathrm{Hn}(|f(a_i) - f(b_i)|)_{i\in\omega} \le |\mathrm{Hn}(K|a_i - b_i|)_{i\in\omega}| \\ &= K\mathrm{Hn}(|a_i - b_i|)_{i\in\omega} = |\alpha - \beta| \end{aligned}$$

because $|f(a_i) - f(b_i)| \le K|a_i - b_i|$ for all $i = 1, 2, 3, \ldots$.

Corollary 3.1.4 *If f is a contraction, then its extrafunction extension F is also a contraction.*

Proposition 3.1.6 *Operations of addition, subtraction, and multiplication by real numbers are correctly defined for real general extrafunctions.*

Indeed, taking two real general extrafunctions F and G, we define $(F + G) \times (\alpha) = F(\alpha) + G(\alpha)$ for any hypernumber α from $\boldsymbol{R}_\omega$. As addition of hypernumbers is uniquely defined, this formula correctly defines addition for general extrafunctions.

In a similar way, we define $(F - G)(\alpha) = F(\alpha) - G(\alpha)$ for any hypernumber α from $\boldsymbol{R}_\omega$. As subtraction of hypernumbers is uniquely defined, this formula correctly defines subtraction for general extrafunctions.

For a real number a, we define $aF(\alpha) = F(a\alpha)$. This formula correctly defines multiplication of real general extrafunctions by real numbers.

Proposition is proved.

As all real hypernumbers form a linear space over $\boldsymbol{R}$, the same is true for real general extrafunctions.

Theorem 3.1.2 *All real general extrafunctions form a linear space over* $\boldsymbol{R}$.

Proof is left as an exercise.

3.1.2 Norm-Based Extrafunctions

Here we consider extrafunctions defined only for real numbers. The simplest class of such extrafunctions consists of restricted real pointwise extrafunctions, which form a subset of all general real extrafunctions.

Definition 3.1.5 A partial mapping $f : \boldsymbol{R} \to \boldsymbol{R}_\omega$ is called a *restricted real pointwise extrafunction*.

We denote by $F(\boldsymbol{R}, \boldsymbol{R}_\omega)$ the set of all restricted real pointwise extrafunctions.

Restricted real pointwise extrafunctions correspond to the topology of pointwise convergence in the space of all real functions. Other topologies on the space of all real functions allow us to define other classes of extrafunctions. To build these classes, we use norms and seminorms (cf. Appendix). Taking a set **F** of real functions, it is possible to define different norms, seminorms, metrics, and topologies in **F**. Note that norms and seminorms in function spaces are functionals. Here are some examples of such normed function spaces.

Example 3.1.4 Let us take **F** equal to the space $C[a, b]$ of all continuous functions on the interval $[a, b]$. The norm in this space is defined by the following formula

$$||f||_\infty = \max\{|f(x)|; x \in [a, b]\}$$

Example 3.1.5 Let us take **F** equal to the space $L^1(a, b)$ of all integrable functions on the interval $[a, b]$. The norm in this space is defined by the following formula

$$||f||_{L^1} = \int_a^b |f(x)|\,\mathrm{d}x$$

Example 3.1.6 Let us take **F** equal to the space $L^2(a, b)$ of all square-integrable functions on the interval $[a, b]$. The norm in this space is defined by the following formula

$$||f||_{L^2} = \left[\int_a^b |f(x)|^2\,\mathrm{d}x\right]^{1/2}$$

Example 3.1.7 Let us take **F** equal to $L^p(a, b)$ is the space of all p-integrable $(p \geq 1)$ functions on the interval $[a, b]$. The norm in this space is defined by the following formula

$$||f||_{L^p} = \left[\int_a^b |f(x)|^p\,\mathrm{d}x\right]^{1/p}$$

Example 3.1.8 Let us take **F** equal to $B[a, b]$ is the space of all bounded continuous functions on the interval $[a, b]$. The norm in this space is defined by the following formula

$$||f||_\infty = \sup\{|f(x)|; x \in [a, b]\}$$

Example 3.1.9 Let us take **F** equal to the Sobolev space $H(a, b)$ of all differentiable functions on the interval $[a, b]$ such that they are square integrable and their derivatives are square integrable. The norm in this space is defined by the following formula

$$||f||_{L^2} = ||f||_{L^2} + ||f'||_{L^2} = \left[\int_a^b |f(x)|^2\mathrm{d}x\right]^{1/2} + \left[\int_a^b |f'(x)|^2\mathrm{d}x\right]^{1/2}$$

To define norm-based extrafunctions, we consider a set **F** of real functions, a set Q of seminorms in **F** and the set $\mathbf{F}^\omega$ of all sequences of functions from **F**.

Definition 3.1.6 For arbitrary sequences $\boldsymbol{f} = (f_i)_{i\in\omega}$ and $\boldsymbol{g} = (g_i)_{i\in\omega}$ of real functions, $\boldsymbol{f} \approx_Q \boldsymbol{g}$ means that $\lim_{i\to\infty} q(f_i - g_i) = 0$ for any $q \in Q$

Lemma 3.1.1 *The relation* $\approx_Q$ *is an equivalence relation in* $\mathbf{F}^\omega$.

Proof By definition, this relation is reflexive. Besides, it is symmetric because $q(f - g) = q(g - f)$ for any seminorm q and we need only to show that the relation $\approx_Q$ is transitive. Taking three sequences $\boldsymbol{f} = (f_i)_{i\in\omega}, \boldsymbol{g} = (g_i)_{i\in\omega}$ and

$\boldsymbol{h} = (h_i)_{i\in\omega}$ of real functions such that $\boldsymbol{f} \approx_Q g$ and $\boldsymbol{g} \approx_Q h$, for any seminorm q from Q, we have

$$\lim_{i\to\infty} q(f_i - g_i) = 0$$

and

$$\lim_{i\to\infty} q(g_i - h_i) = 0$$

By properties of seminorms and limits, we have

$$\begin{aligned} 0 &\leq \lim_{i\to\infty} q(f_i - h_i) = \lim_{i\to\infty} q(f_i - g_i + g_i - h_i) \\ &\leq \lim_{i\to\infty} q(f_i - g_i) + \lim_{i\to\infty} q(g_i - h_i) = 0 + 0 = 0 \end{aligned}$$

Consequently,

$$\lim_{i\to\infty} q(f_i - h_i) = 0$$

i.e., $\boldsymbol{f} \approx_Q \boldsymbol{h}$.

Lemma is proved.

Definition 3.1.7 Classes of the equivalence relation $\approx_Q$ are called *Q-based real extrafunctions represented in* **F** and their set is denoted by $\mathbf{E}^{\mathbf{F}}_{\omega Q}$.

In such a way, any sequence $\boldsymbol{f} = (f_i)_{i\in\omega}$ of real functions from **F** determines (and represents) a Q-based real extrafunction $F = \mathrm{EF}_Q(f_i)_{i\in\omega}$, and this sequence is called a *defining sequence* or *representing sequence* or *representation* of the extrafunction $F = \mathrm{EF}_Q(f_i)_{i\in\omega}$. There is also a natural projection $\pi_Q : \mathbf{F}^\omega \to \mathbf{E}^{\mathbf{F}}_{\omega Q}$, where $\pi_Q((f_i)_{i\in\omega}) = \mathrm{EF}_Q(f_i)_{i\in\omega}$ for any sequence $\boldsymbol{f} = (f_i)_{i\in\omega} \in \mathbf{F}^\omega$.

We see that the construction of norm-based extrafunctions is similar to the construction of hypernumbers.

Lemma 3.1.2 *If* **F** *and* **H** *are sets of real functions and* $\mathbf{F} \subseteq \mathbf{H}$, *then for any system of seminorms P, we have* $\mathbf{E}^{\mathbf{F}}_{\omega P} \subseteq \mathbf{E}^{\mathbf{H}}_{\omega P}$.

Proof is left as an exercise.

An important question is when the space **F** is a natural subspace of the hyperspace $\mathbf{E}^{\mathbf{F}}_{\omega Q}$.

Definition 3.1.8 A system Q of seminorms in **F** (weakly) *separates* **F** if for any elements f and g from $\boldsymbol{L}$, there is a seminorm q from Q such that $q(f) \neq q(g)(q(f - g) \neq 0)$.

Lemma 3.1.3 *Any norm in* **F** *separates* **F**.

It is possible to define a mapping $\mu_Q : \boldsymbol{F} \to \mathbf{E}^{\mathbf{F}}_{\omega Q}$ by the formula $\mu_Q(f) = \mathrm{EF}_Q(f_i)_{i\in\omega}$ where $f_i = f$ for all $i = 1, 2, 3, \ldots$.

Proposition 3.1.7

a) μ_Q *is an injection (monomorphism of linear spaces when* **F** *is a linear space) if the system Q of seminorms separates* **F**.

b) μ_Q *is an injection (monomorphism of linear spaces when* **F** *is a linear space) if and only if the system Q of seminorms weakly separates* **F**.

Proof
Sufficiency
If f and g are different elements from F and the system Q of seminorms in F Weakly separates F, then there is a seminorm q from Q such that $q(f-g) \neq 0$. Then by definition, $\mu_Q(f) \neq \mu_Q(g)$ and μ_Q is an injection.
If f and g are different elements from F and the system Q of seminorms in F separates F, then there is a seminorm q from Q such that $q(f) \neq q(g)$. Consequently, by properties of seminorms, $q(f-g) \neq 0$ i.e., the system Q of seminorms in F weakly separates F. Thus, as it is proved above, μ_Q is a monomorphism.

Necessity
If the system Q of seminorms does not weakly separate **F**, then for some elements f and g from **F**, $q(f-g) = 0$ for all seminorms q from Q. It means, by Definition 3.1.6, that $(f_i)_{i\in\omega} \approx_Q (f_i)_{i\in\omega}$ where $f_i = f$ and $g_i = g$ for all $i = 1, 2, 3, \ldots$.

Thus, $\mu_Q(f) = \mu_Q(g)$ and μ_Q is not an injection. So, by the Law of Contraposition for propositions (cf., for example, (Church, 1956)), the condition from Proposition 3.1.7 is necessary.

Proposition is proved.

Definition 3.1.9
(a) A sequence $\{f_i;\ i = 1, 2, 3, \ldots\}$ of functions *Q-converges* to a function f, i.e., $\lim^Q_{i\to\infty} f_i = f$, if $\lim_{i\to\infty} q(f - f_i) = 0$ for all $q \in Q$ for all $q \in Q$. In this case, f is called a *Q-limit* of functions f_i.
(b) A sequence $\{f_i;\ i = 1, 2, 3, \ldots\}$ of functions *uniformly Q-converges* to a function f if for any $\varepsilon > 0$, there is a natural number n such that if $q \in Q$ and $i > n$, then $q(f - f_i) < \varepsilon$. In this case, f is called a *uniform Q-limit* of functions f_i.

The concept of Q-convergence generalizes the concept of conventional convergence and concept of Q-limit generalizes the concept of conventional limit.

Proposition 3.1.8 *If* **H** *and* **G** *are sets of real functions and any function f from* **H** *is a uniform* Q*-limit of functions from* **G**, *then any extrafunction represented in* **H** *is also represented in* **G**.

Proof Let us take an extrafunction $F = \mathrm{EF}_Q(f_i)_{i\in\omega}$ where all $f_i \in \mathbf{H}$. As any function f from **H** is a uniform Q-limit of functions from **G**, we see that for any f_i, there is a function h_i from **G** and natural number n such that if $q \in Q$ and $i > n$, then $q(f - f_i) < 1/i$. By Definitions 3.1.6 and 3.1.7, it means that $F = \mathrm{EF}_Q(h_i)_{i\in\omega}$, i.e., F is represented in **G**.

Proposition is proved.

Relations between systems of seminorms induce relations between corresponding classes of norm-based extrafunctions. Let us consider some of them, taking two systems of seminorms $Q = \{q_i; i \in I\}$ and $P = \{p_j; j \in J\}$ in a space **F**. By the definition of $\mathbf{E}^{\mathbf{F}}_{\omega P}$ and $\mathbf{E}^{\mathbf{F}}_{\omega Q}$, there are natural projections $\pi_P : \mathbf{F}^\omega \to \mathbf{E}^{\mathbf{F}}_{\omega P}$ and $\pi_Q : \mathbf{F}^\omega \to \mathbf{E}^{\mathbf{F}}_{\omega Q}$.

Definition 3.1.10

(a) *P dominates Q* in X (it is denoted by $P > Q$) if

$$\forall q \in Q \exists p \in P \exists k_q \in \mathbf{R}^{++} \forall x \in \mathbf{F}(q(x) \leq k_q \cdot p(x))$$

(b) *P regularly dominates Q* in X (it is denoted by $P \lhd Q$) if

$$\forall q \in Q \exists p \in P \forall x \in \mathbf{F}(q(x) \leq p(x))$$

Lemma 3.1.4 *If* $\mathrm{P} > \mathrm{Q}$*, then for any sequences* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *from* $\mathbf{F}^{\omega}$*, we have*

$$\boldsymbol{f} \approx_P \boldsymbol{g} \quad \text{implies} \quad \boldsymbol{f} \approx_Q \boldsymbol{g}$$

Corollary 3.1.5 *If* $\mathrm{P} \rhd \mathrm{Q}$*, then for any sequences* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *from* $\mathbf{F}^{\omega}$*, we have*

$$\boldsymbol{f} \approx_P \boldsymbol{g} \quad \text{implies} \quad \boldsymbol{f} \approx_Q \boldsymbol{g}$$

These properties of function sequences imply corresponding properties of spaces of functions.

Theorem 3.1.3 *If* $\mathrm{P} > \mathrm{Q}$*, then there is a natural projection* $\pi : \mathbf{E}^{\mathbf{F}}_{\omega P} \to \mathbf{E}^{\mathbf{F}}_{\omega Q}$ *for which the following equality is valid* $\pi \circ \pi_P = \pi_Q$.

Corollary 3.1.6 *If* $\mathrm{P} \rhd \mathrm{Q}$*, then there is a natural projection* $\pi : \mathbf{E}^{\mathbf{F}}_{\omega P} \to \mathbf{E}^{\mathbf{F}}_{\omega Q}$ *for which the following equality is valid* $\pi \circ \pi_P = \pi_Q$.

Different norms and seminorms allow us to build various classes of norm-based extrafunctions. Here we consider several sets Q of seminorms, building important types of extrafunctions, which are related to types of extrafunctions, distributions and other generalized functions studied before.

Type 1 In the space $F(\boldsymbol{R})$ of all real functions, it is possible to define the seminorm $q_{\text{pt}x}$ using an arbitrary real number x and the following formula

$$q_{\text{ptx}}(f) = |f(x)|$$

Indeed, properties of the absolute value imply that this functional satisfies both Conditions N2 and N3 from the definition of a seminorm (cf. Appendix).

We define $Q_{\text{pt}} = \{q_{\text{ptx}}; x \in \boldsymbol{R}\}$. Definitions 3.1.6 and 3.1.7 show how this set of seminorms determines the equivalence relation $\approx_{\text{pt}}$ in the space of all sequences of real functions and defines Q_{pt}-based real extrafunctions represented in $F(\boldsymbol{R})$. A sequence $\boldsymbol{f} = (f_i)_{i\in\omega}$ of real functions determines (and represents) a Q_{pt}-based real extrafunction $F = \mathrm{EF}_{\mathrm{Qpt}}(f_i)_{i\in\omega}$, which is also denoted by $F = \mathrm{Ep}(f_i)_{i\in\omega}$.

Proposition 3.1.9 *The set* $\mathbf{E}^{F(\boldsymbol{R})}_{\omega Q_{\text{pt}}}$ *of all* Q_{pt}*-based extrafunctions is isomorphic to the set* $F(\boldsymbol{R}, \boldsymbol{R}_{\omega})$ *of all restricted real pointwise extrafunctions.*

Proof is left as an exercise.

Note that the set $\mathbf{E}^{F(\boldsymbol{R})}_{\omega Q_{\text{pt}}}$ does not coincide with the set $F(\boldsymbol{R}, \boldsymbol{R}_\omega)$ because elements of the first set are classes of functions, while elements of the second set are mappings.

Proposition 3.1.9 implies that it is possible to represent restricted real pointwise extrafunctions by sequences of ordinary real functions. Namely, a sequence $(f_i)_{i\in\omega}$ of real functions f_i represents a restricted pointwise extrafunction $F = \text{EF}_{Q\text{pt}}(f_i)_{i\in\omega}$ such that $F(x = \text{Hn}(f_i(x))_{i\in\omega}$ for all x. However, in some cases, it is possible to find a representation with additional useful properties.

Definition 3.1.11 A restricted real pointwise extrafunction $F : \boldsymbol{R} \to \boldsymbol{R}_\omega$ is *continuously represented* if there is a sequence $(f_i)_{i\in\omega}$ of continuous real functions f_i such that $F = \text{EF}_{Q_{\text{pt}}}(f_i)_{i\in\omega}$.

For instance, the extrafunction defined by the formula $F(x) = \text{Hn}(nx)_{n\in\omega}$ is continuously represented.

Proposition 3.1.10 *For any continuously represented restricted pointwise extrafunction* F*, there is a sequence* $(g_i)_{i\in\omega}$ *of bounded continuous real functions* g_i *such that* $F = \text{EF}_{Q\text{pt}}(f_i)_{i\in\omega}$.

Proof Let $F = \text{EF}_{Q\text{pt}}(f_i)_{i\in\omega}$ and all f_i are continuous real functions. Then we consider the sequence of intervals $[-k, k]$ with $k = 1, 2, \ldots, n, \ldots$. This makes it possible to define a function $g_k(x)$ equal to the function $f_k(x)$ when x belongs to the interval $[-k, k]$, equal to $f_k(-k)$ when $x < -k$, and equal to $f_k(k)$ when $x > k$. It means that the function $g_k(x)$ coincides with the function $f_k(x)$ inside the interval $[-k, k]$, is equal to the same value $f_k(-k)$ for all $x \leq -k$ and to the same value $f_k(k)$ for all $x \geq k$. As a function continuous in $\boldsymbol{R}$ is bounded in any finite interval, all $g_i(x)$ are bounded functions and it is possible to check that $F = \text{EF}_{Q\text{pt}}(f_i)_{i\in\omega}$.

Proposition is proved.

Remark 3.1.1 Not all restricted pointwise real extrafunctions and even not all real functions can be continuously represented. To show this, let us consider the characteristic function h of the set of all rational numbers in the interval [0, 1]. It is equal to 1 for all rational numbers from the interval [0, 1] and is equal to 0 for all other real numbers.

In more detail, real pointwise extrafunctions are studied in Burgin (2002) and applied to the path integral in Burgin (2008/2009). Complex pointwise extrafunctions are studied and applied to differential equations in Burgin and Ralston (2004) and Burgin (2010).

Type 2 Let us consider a class **K** of functions such that for any function $f(x)$ from the class **F** and any function $g(x)$ from **K**, the integral $\int f(x)g(x)\, \mathrm{d}x$ exists. If functions from both classes are defined in $\boldsymbol{R}$, then the integral is taken over the whole of $\boldsymbol{R}$, while if functions from both classes are defined in an interval $[a, b]$, then the integral is taken over the interval $[a, b]$. For instance, **K** consists of all continuous functions with the compact support and **F** consists of all continuous functions. For simplicity, we consider the Riemann integral although it is possible to use other kinds of integrals, e.g., Lebesgue integral, Stieltjes integral, Perron integral, Denjoy integral, or gauge integral, for this purpose. For instance, taking integrals with respect to a system of measures on intervals of real numbers, we come to the concept of a real measure-wise extrafunction introduced and studied in Burgin (2002). In all cases, extended distributions are special kinds of extrafunctions.

Then it is possible to define the seminorm q_g in **F** using an arbitrary function $g(x)$ from **K** and the following formula

$$q_g(f) = \left| \int f(x)g(x)\,\mathrm{d}x \right|$$

Indeed, q_g satisfies Condition N3 from the definition of a seminorm because the integral is a linear functional and the absolute value satisfies the triangle inequality. Besides, it satisfies condition N2 from the definition of a seminorm because the absolute value of a real number is a norm.

We define $Q_{\mathbf{K}} = \{q_g; g \in \mathbf{K}\}$. Definitions 3.1.6 and 3.1.7 show how this set of seminorms determines the equivalence relation $\approx_{\mathbf{K}}$ in the space of all sequences of real functions from **F**, as well as $Q_{\mathbf{K}}$-based real extrafunctions represented in **F**. According to the tradition (cf., for example, Rudin 1973), we call **K** the *set of test functions*. A sequence $f = (f_i)_{i\in\omega}$ of functions from **F** determines (and represents) a $Q_{\mathbf{K}}$-based real extrafunction $F = \mathrm{EF}_{Q_{\mathbf{K}}}(f_i)_{i\in\omega}$ represented in **F**.

The classes of sequences of continuous real functions equivalent with respect to **K** are called *real extended distributions* with respect to **K**. We denote the class of all real extended distributions with respect to **K** by $\mathbf{D}_{\mathbf{K}\omega}$, the expression $\mathrm{Ed}_{\mathbf{K}}(f_i)_{i\in\omega}$ denotes the real extended distribution with respect to **K** that is defined by the sequence of continuous functions $(f_i)_{i\in\omega}$, and the class of all such extended distributions is denoted by $\mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\mathbf{K}}}$.

In a natural way, extended distributions are connected to conventional distributions or generalized functions, which appeared in the following way. In classical analysis, continuous functions do not always have derivatives. To remedy this deficiency and to be able to solve various differential equations from physics, some distributions, such as Dirac delta-function $\delta(x)$, were used by physicists. Later the concept of a *distribution* was rigorously defined by mathematicians and theory of distributions was developed. There are different equivalent ways to define distributions. The definition of a distribution as a functional was historically the first (Schwartz 1950/1951). Another approach is called sequential because in it, distributions are defined as classes of equivalent sequences of ordinary functions (Antosik et al 1973; Liverman 1964). Thus, it is natural that distributions are closely related to extrafunctions. Here we explain this relation.

Definition 3.1.12 A sequence $\{f_n; n \in \omega\}$ of real functions *converges almost uniformly* in $\boldsymbol{R}$ if it uniformly converges on every interval $[a, b]$.

Definition 3.1.13 A sequence $\{f_n; n \in \omega\}$ of continuous real functions is called *fundamental* if for any interval $I = [a, b]$, there are a sequence $\{F_n; n \in \omega\}$ and a natural number k that satisfy the following conditions:

1. $F_n^{(k)}(x) = f_n(x)$ for all $x \in I$ and $n \in \omega$ where $F_n^{(k)}(x) = (\mathrm{d}^k F_n(x))/\mathrm{d}x^k$ is the kth-order derivative of the function $F_n(x)$.
2. The sequence $\{F_n; n \in \omega\}$ converges almost uniformly in $\boldsymbol{R}$.

Definition 3.1.14 Two fundamental sequences $\{f_n; n \in \omega\}$ and $\{g_n; n \in \omega\}$ are *equivalent* if the sequence $\{f_1, g_1, f_2, g_2, f_3, \ldots\}$ is fundamental, i.e., for any interval $I = [a, b]$, there are sequences $\{F_n; n \in \omega\}$ and $\{G_n; n \in \omega\}$ and a natural number k such that $F_n^{(k)}(x) = f_n(x)$, $G_n^{(k)}(x) = g_n(x)$ for all $x \in I$ and $x \in \omega$ and sequences $\{F_n; n \in \omega\}$ and $\{G_n; n \in \omega\}$ converge almost uniformly in $\boldsymbol{R}$ to the same function.

Definition 3.1.15 A class of equivalent fundamental sequences is called a *distribution.*

If $\{f_n; n \in \omega\}$ is a fundamental sequence, then $D\{f_n; n \in \omega\}$ denotes the distribution that contains $\{f_n; n \in \omega\}$, i.e., the class of all sequences equivalent to the sequence $\{f_n; n \in \omega\}$. The set of all distributions is denoted by D'.

Let us consider the space **CD** of all functions from $C^\infty(\boldsymbol{R})$ that have the compact support (Schwartz 1950/1951).

Theorem 3.1.4 *There is a subspace* **D** *of the space* $\mathbf{D}_{\mathbf{CD}\omega}$, *which is isomorphic to* D'.

Proof As the space $\mathbf{D}_{\mathbf{CD}\omega}$ is a set of all equivalent classes of function sequences, we can take the subspace of this space generated by fundamental sequences as **D**. Using definitions, it is possible to show that two fundamental sequences $\{f_n; n \in \omega\}$ and $\{g_n; n \in \omega\}$ are equivalent as extended distributions if and only if they are equivalent in the sense of Definition 3.1.13 This gives us the necessary isomorphism between spaces **D** and D'.

Theorem 3.1.4 shows that distributions are a special case of extrafunctions, namely, of extended distributions, in a similar way as real numbers are a special case of real hypernumbers.

In more detail, real extended distributions are studied in Burgin (2004) and applied to the path integral in Burgin (2008/2009). Complex extended distributions are studied and applied to differential equations in Burgin and Ralston (2004) and Burgin (2010).

Type 3 Let us consider the space $C(\boldsymbol{R})$ of all functions continuous in $\boldsymbol{R}$. Then it is possible to define the seminorm $q_{\max[a,\, b]}$ in $C(\boldsymbol{R})$ using an arbitrary interval $[a, b]$ and the following formula

$$q_{\max[a,b]}(f) = \max\{|f(x)|; x \in [a, b]\}$$

Indeed, we have

$$\begin{aligned} q_{\max[a,b]}(f+g) &= \max\{|f(x)+g(x)|; x \in [a,b]\} \leq \max\{(|f(x)+|g(x)|); x \\ &\in [a,b]\} = \max\{|f(x)|; x \in [a,b]\} + \max\{|g(x)|; x \in [a,b]\} \\ &= q_{\max[a,b]}(f) + q_{\max[a,b]}(g) \end{aligned}$$

and if a is a real number, then

$$q_{\max[a,b]}(af) = \max\{|af(x)|; x \in [a,b]\} = \max\{|a||f(x)|; x \in [a,b]\}$$
$$= |a| \max\{|f(x)|; x \in [a,b]\} = |a| q_{\max[a,b]}(f)$$

We define $Q_{\text{comp}} = \{q_{\max[a,b]}; a, b \in \boldsymbol{R}\}$. Definitions 3.1.6 and 3.1.7 show how this set of seminorms determines the equivalence relation $\approx_{\text{comp}}$ in the space of all sequences of continuous in $\boldsymbol{R}$ functions, as well as continuously represented Q_{comp}-based real extrafunctions. A sequence $\boldsymbol{f} = (f_i)_{i\in\omega}$ of continuous functions determines (and represents) a Q_{comp}-based real extrafunction $F = \text{EF}_{Q_{\text{comp}}}(f_i)_{i\in\omega}$, which is also denoted by $F = \text{Ec}(f_i)_{i\in\omega}$ and called a *real continuously represented compactwise extrafunction*. $\text{Comp}(\boldsymbol{R}, \boldsymbol{R}_\omega)$ is the set/space of all real continuously represented compactwise extrafunctions.

Continuously represented compactwise extrafunctions are also related to distributions. For instance, as it proved in Burgin (2001), there is the linear subspace Comp D of $\text{Comp}(\boldsymbol{R}, \boldsymbol{R}_\omega)$ and a linear projection $p : \text{Comp D} \to D'$.

In more detail, real continuously represented compactwise extrafunctions are studied in Burgin (2001, 2002) and applied to the path integral in Burgin (2008/2009). Complex continuously represented compactwise extrafunctions are studied and applied to differential equations in Burgin and Ralston (2004) and Burgin (2010).

Type 4 Let us consider the space $BI(\boldsymbol{R})$ of all real functions bounded in each interval $[a, b]$. Then it is possible to define the seminorm $q_{\sup[a,\, b]}$ in $BI(\boldsymbol{R})$ using an arbitrary interval $[a, b]$ and the following formula

$$q_{\sup[a,b]}(f) = \sup\{|f(x); x \in [a,b]\}$$

Indeed, we have

$$q_{\sup[a,b]}(f+g) = \sup\{|f(x)+g(x)|; x \in [a,b]\} \leq \sup\{(|f(x)|+|g(x)|); x \in [a,b]\} = \sup\{|f(x)|; x \in [a,b]\} + \sup\{|g(x)|; x \in [a,b]\}$$
$$= q_{\sup[a,b]}(f) + q_{\sup[a,b]}(g)$$

and if a is a real number, then

$$q_{\sup[a,b]}(af) = \sup\{|af(x)|; x \in [a,b]\} = \sup\{|a||f(x)|; x \in [a,b]\}$$
$$= |a| \sup\{|f(x)|; x \in [a,b]\} = |a| q_{\sup[a,b]}(f)$$

We define $Q_{\text{cp}} = \{q_{\sup[a,b]}; a, b \in \boldsymbol{R}\}$. Definitions 3.1.6 and 3.1.7 show how this set of seminorms determines the equivalence relation $\approx_{\text{cp}}$ in the space of all sequences of bounded in $\boldsymbol{R}$ functions, as well as continuously represented Q_{cp}-based real extrafunctions. A sequence $\boldsymbol{f} = (f_i)_{i\in\omega}$ of bounded functions determines (and represents) a $Q_{\rm cp}$-based real extrafunction $F = \mathrm{EF}_{Q_{\rm cp}}(f_i)_{i\in\omega}$, which is also denoted by $F = \mathrm{Ebc}(f_i)_{i\in\omega}$ and called a real boundedly represented compactwise extrafunction.

Theorem 3.1.3 allows us to find relations between the four introduced types of functions.

Proposition 3.1.11 $Q_{\rm comp} \triangleright Q_{\rm pt}$ *in the set C(**R**) of all continuous real functions.*

Proof is left as an exercise.

Corollary 3.1.7 *There is a projection* $\sigma_{\rm cont} : \mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\rm comp}} \to \mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\rm pt}}$.

Proposition 3.1.12 $Q_{\rm comp} = Q_{\rm cp}$ *in the set C(**R**) of all continuous real functions.*

Proof is left as an exercise.

Corollary 3.1.8 $\mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\rm comp}} = \mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\rm cp}}$.

Proposition 3.1.13 $Q_{\rm cp} \triangleright Q_{\rm pt}$ *in the set* BI*(**R**) of all real functions that are bounded in each finite interval.*

Proof is left as an exercise.

Corollary 3.1.9 *There is a projection* $\tau_{\rm fbd} : \mathbf{E}^{BI(\boldsymbol{R})}{}_{\omega Q_{\rm cp}} \to \mathbf{E}^{BI(\boldsymbol{R})}{}_{\omega Q_{\rm pt}}$.

Proposition 3.1.14 $Q_{\rm comp} > Q_{\mathbf{K}}$ *in the set* C*(**R**) if all functions in* **K** *have a compact support.*

Proof is left as an exercise.

Corollary 3.1.10 *There is a projection* $\tau_{\rm cst} : \mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\rm comp}} \to \mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\mathbf{K}}}$.

Proposition 3.1.15 $Q_{\rm cp} > Q_{\mathbf{K}}$ *in the set* BI*(**R**) if all functions in* **K** *have a compact support.*

Proof is left as an exercise.

Corollary 3.1.11 *There is a projection* $\tau_{\rm cst} : \mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\rm cp}} \to \mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\mathbf{K}}}$.

3.2 Algebraic Properties

Let us assume that the class **F** of real functions with a system Q of seminorms is closed with respect to addition and/or subtraction of functions. For instance, **F** is an abelian group if it is the set $C(\boldsymbol{R})$ of all continuous real functions. Then addition and subtraction of functions induce corresponding operations in sets of extrafunctions represented in **F**.

Proposition 3.2.1 *Operations of addition and/or subtraction are correctly defined for Q-based real extrafunctions represented in* **F**.

Proof Let us take two Q-based real extrafunctions $F = \mathrm{EF}_Q(f_i)_{i\in\omega}$ and $G = \mathrm{EF}_Q(g_i)_{i\in\omega}$. We define $F + G = \mathrm{EF}_Q(f_i + g_i)_{i\in\omega}$. To show that this is an operation with extrafunctions, it is necessary to prove that $F + G$ does not depend on the choice of a representing sequences $(f_i)_{i\in\omega}$ and $(g_i)_{i\in\omega}$ for the extrafunctions F and G. To do this, let us take another sequence $(h_i)_{i\in\omega}$ that represents F and show that $F + G = \mathrm{EF}_Q(h_i + g_i)_{i\in\omega}$. Note that all three sequences $(f_i)_{i\in\omega}$, $(h_i)_{i\in\omega}$, and $(g_i)_{i\in\omega}$ belong to $\mathbf{F}^{\omega}$.

In this case, for any seminorm q from Q, we have

$$\lim_{i\to\infty} q((f_i + g_i) - (h_i + g_i)) = 0$$

because $\lim_{i\to\infty} q(f_i - h_i) = 0$. Thus, addition is correctly defined for Q-based real extrafunctions represented in $\mathbf{F}$.

The proof for the difference of two Q-based real extrafunctions is similar.

Proposition is proved.

The construction of addition and subtraction of Q-based real extrafunctions represented in $\mathbf{F}$ implies the following result.

Proposition 3.2.2 *Any identity that involves only operations of addition and/or subtraction and is valid for real functions is also valid for Q-based real extrafunctions represented in* $\mathbf{F}$.

Corollary 3.2.1 *If* $\mathbf{F}$ *is a semigroup, then* $\mathbf{E}^{\mathbf{F}}{}_{\omega Q}$ *is also a semigroup.*

Corollary 3.2.2 *If* $\mathbf{F}$ *is an abelian group, then* $\mathbf{E}^{\mathbf{F}}{}_{\omega Q}$ *is also an Abelian group.*

Let us assume that the class $\mathbf{F}$ is closed with respect to multiplication by real numbers, i.e., if a is a real number and $f \in \mathbf{F}$, then the product af also belongs to $\mathbf{F}$. Then this operation induces the corresponding operation in sets of represented in $\mathbf{F}$ extrafunctions.

Proposition 3.2.3 *Multiplication by real numbers is correctly defined for Q-based real extrafunctions represented in* $\mathbf{F}$.

Proof Let us take a Q-based real extrafunction $F = \mathrm{EF}_Q(f_i)_{i\in\omega}$ and a real number a. We define $aF = \mathrm{EF}_Q(af_i)_{i\in\omega}$. Let us take another sequence $(h_i)_{i\in\omega}$ that represents F and show that $aF = \mathrm{EF}_Q(ah_i)_{i\in\omega}$. Indeed, for any seminorm q from Q, we have

$$\lim_{i\to\infty} q(af_i - ah_i) = \lim_{i\to\infty} |a| q(f_i - h_i) = |a| \cdot \lim_{i\to\infty} q(f_i - h_i) = 0$$

because $\lim_{i\to\infty} q(f_i - h_i) = 0$ and q satisfies Condition N2. Thus, multiplication by real numbers is correctly defined for Q-based real extrafunctions represented in $\mathbf{F}$.

Proposition is proved.

Proposition 3.2.4 *Any identity that is valid for real functions and involves only operations of addition and/or multiplication by real numbers is also valid for Q-based real extrafunctions represented in* $\mathbf{F}$.

Proof is left as an exercise.

In particular, Proposition 3.2.4 gives the following identities for extrafunctions.

1. $$F(x) + G(x) = G(x) + F(x)$$
2. $$F(x) + (G(x) + H(x)) = (F(x) + G(x)) + H(x)$$
3. $$a(F(x) + G(x)) = aF(x) + aG(x)$$
4. $$(a + b)F(x) = aF(x) + bF(x)$$
5. $$a(bF(x)) = (ab)F(x)$$
6. $$1 \cdot F(x) = F(x)$$

Propositions 3.2.1, 3.2.2, and 3.2.4 imply the following result.

Theorem 3.2.1 *If* **F** *is a linear space over the field of real numbers* $\boldsymbol{R}$*, then* $\mathbf{E}^{\mathbf{F}}{}_{\omega Q}$ *is also a linear space over the field of real numbers* $\boldsymbol{R}$.

Corollary 3.2.3 *The set of all restricted extrafunctions is a linear space over the field of real numbers* $\boldsymbol{R}$.

Corollary 3.2.4 *The set of all continuously represented restricted real pointwise extrafunctions is a linear space over the field of real numbers* $\boldsymbol{R}$.

Corollary 3.2.5 *The set of all real compactwise extrafunctions is a linear space over the field of real numbers* $\boldsymbol{R}$.

Corollary 3.2.6 *The set of all real extended distributions is a linear space over the field of real numbers* $\boldsymbol{R}$.

Theorems 3.1.3 and 3.2.1 imply the following result.

Theorem 3.2.2 *If* $P > Q$ *and* **F** *is a linear space over* $\boldsymbol{R}$*, then there is a linear mapping* π *of the linear space* $\mathbf{E}^{\mathbf{F}}{}_{\omega P}$ *over* $\boldsymbol{R}$ *onto the linear space* $\mathbf{E}^{\mathbf{F}}{}_{\omega Q}$ *over* $\boldsymbol{R}$.

Corollary 3.2.7 *There is a linear mapping* σ_{cont} *of the linear space* $\mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\mathrm{comp}}}$ *onto the linear space* $\mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\mathrm{pt}}}$.

Corollary 3.2.8 *There is a linear mapping* τ_{fbd} *of the linear space* $\mathbf{E}^{BI(\boldsymbol{R})}{}_{\omega Q_{\mathrm{cp}}}$ *onto the linear space* $\mathbf{E}^{BI(\boldsymbol{R})}{}_{\omega Q_{\mathrm{pt}}}$.

Corollary 3.2.9 *There is a linear mapping* τ_{cst} *of the linear space* $\mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\mathrm{comp}}}$ *onto the linear space* $\mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\mathbf{K}}}$.

Let us assume that the class **F** is closed with respect to multiplication by functions from a class **H**, i.e., if g is a function from **H** and $f \in \mathbf{F}$, then $g \cdot f \in \mathbf{F}$, and for each q from Q, **H** is a seminormed ring and **F** is a seminormed module over **H**. Then multiplication by functions from **H** induces the corresponding operation in sets of extrafunctions represented in **F**.

Proposition 3.2.5 *Multiplication by functions from* **H** *is correctly defined for* Q*-based real extrafunctions represented in* **F**.

Proof Let us take a Q-based real extrafunction $F = \mathrm{EF}_Q(f_i)_{i\in\omega}$ and a function g from **H**. We define $g \cdot F = \mathrm{EF}_Q(g \cdot f_i)_{i\in\omega}$. Let us take another sequence $(h_i)_{i\in\omega}$ that represents F and show that $g \cdot F = \mathrm{EF}_Q(g \cdot h_i)_{i\in\omega}$. Note that both sequences $(f_i)_{i\in\omega}$ and $(h_i)_{i\in\omega}$ belong to $\mathbf{F}^\omega$. By conditions from the theorem, for each q from Q, **H** is a seminormed ring, **F** is a seminormed module over **H** and thus, q satisfies Condition N5. Consequently, for any seminorm q from Q, we have

$$\lim_{i\to\infty} q(g \cdot f_i - g \cdot h_i) \leq \lim_{i\to\infty} q(g) \cdot q((f_i - h_i)) = 0$$

because $\lim_{i\to\infty} q(f_i - h_i) = 0$ and as q is a non-negative function, we have

$$\lim_{i\to\infty} q((g \cdot f_i - g \cdot h_i)) = 0$$

Thus, multiplication by functions from **H** is correctly defined for Q-based real extrafunctions represented in **F**.

Proposition is proved.

Proposition 3.2.5 and 3.2.4 imply the following result.

Theorem 3.2.3 *If the class* **F** *is a* Q*-seminormed module over a* Q*-seminormed ring* **H** *of real functions, then* $\mathbf{E}^{\mathbf{F}}{}_{\omega Q}$ *is a module over the ring* **H**.

As the class of all total real functions is a Q_{pt}-seminormed ring, we have the following result.

Corollary 3.2.10 *The set of all restricted extrafunctions is a module over the ring of all total real functions.*

As the class of all total real functions is a Q_{pt}-seminormed module over the Q_{pt}-seminormed ring of all total continuous real functions, we have the following results.

Corollary 3.2.11 *The set of all continuously represented restricted real pointwise extrafunctions is a module over the ring of all total continuous real functions.*

Corollary 3.2.12 *The set of all real compactwise extrafunctions is a module over the ring of all total continuous real functions.*

Corollary 3.2.13 *The set of all real extended distributions is a module over the ring of all total continuous real functions.*

Theorems 3.1.3 and 3.2.3 imply the following result.

Theorem 3.2.4 *If* $\mathrm{P} > \mathrm{Q}$ *and* **F** *is* $(P\bigcup Q)$*-seminormed module over the* $(P\bigcup Q)$*-seminormed ring* **H***, then there is a homomorphism* π *of the* P*-seminormed module* $\mathbf{E}^{\mathbf{F}}{}_{\omega P}$ *over* **H** *onto the* Q*-seminormed module* $\mathbf{E}^{\mathbf{F}}{}_{\omega Q}$ *over* **H**.

Corollary 3.2.14 *There is a homomorphism* σ_{cont} *of the module* $\mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\text{comp}}}$ *onto the module* $\mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\text{pt}}}$.

Corollary 3.2.15 *There is a homomorphism* τ_{fbd} *of the module* $\mathbf{E}^{BI(\boldsymbol{R})}{}_{\omega Q_{\text{cp}}}$ *onto the module* $\mathbf{E}^{BI(\boldsymbol{R})}{}_{\omega Q_{\text{pt}}}$.

Corollary 3.2.16 *There is a homomorphism* τ_{cst} *of the module* $\mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\text{comp}}}$ *onto the module* $\mathbf{E}^{C(\boldsymbol{R})}{}_{\omega Q_{\mathbf{K}}}$.

As before, we assume that q is a seminorm and Q is a set of seminorms.

Definition 3.2.1 A Q-based real extrafunction $F = \text{EF}_Q(f_i)_{i\in\omega}$ is called *Q-bounded* if for any seminorm q from Q, there is a positive real number c and a representation $(g_i)_{i\in\omega}$ of F, such that $q(g_i)<c$ for almost all $i = 1, 2, 3, \ldots$. The set of all Q-bounded Q-based real extrafunctions represented in $\mathbf{F}$ is denoted by $\mathbf{BE}^{\mathbf{F}}_{\omega Q}$.

Lemma 3.2.1 *For any representation* $(f_i)_{i\in\omega}$ *of a Q-bounded Q-based real extrafunction F and any seminorm q from Q, there is a positive real number d, such that* $q(f_i)<d$ *for almost all* $i = 1, 2, 3, \ldots$.

Proof As F is a Q-bounded Q-based real extrafunction, there is a positive real number c and a representation $(g_i)_{i\in\omega}$ of F, such that $q(g_i)<c$ when $i > m$. As $(f_i)_{i\in\omega}\approx_Q(g_i)_{i\in\omega}$, we have $\lim_{i\to\infty}q(f_i - g_i) = 0$ for any $q \in Q$. Thus, there is a positive real number k, such that $q(f_i - g_i)<k$ when $i > n$.

As q is a seminorm, we have

$$q(f_i) = q((f_i - g_i) + g_i) \leq q(f_i - g_i) + q(g_i)<c + k$$

when $i > \max\{m, n\}$.

Lemma is proved because we can take $d = c + k$.

Corollary 3.2.17 *For any representation* $(f_i)_{i\in\omega}$ *of a Q-bounded Q-based real extrafunction F and any seminorm q from Q, there is a positive real number d, such that* $q(f_i)< d$ *for all* $i = 1, 2, 3, \ldots$.

Proof is left as an exercise.

Corollary 3.2.18 *For any seminorm q from Q, there is a positive real number d, such that for any representation* $(f_i)_{i\in\omega}$ *of a Q-bounded Q-based real extrafunction F, we have* $q(f_i)< d$ *for almost all* $i = 1, 2, 3, \ldots$.

Proof is left as an exercise.

Let us assume that the class $\mathbf{F}$ is closed with respect to multiplication, i.e., if $g,f \in \mathbf{F}$, then $g \cdot f \in \mathbf{F}$, and for each q from Q, $\mathbf{F}$ is a seminormed algebra over $\boldsymbol{R}$. Then multiplication in $\mathbf{F}$ induces the corresponding operation in sets of Q-bounded extrafunctions represented in $\mathbf{F}$.

Proposition 3.2.6 *Multiplication of Q-bounded Q-based real extrafunctions represented in* $\mathbf{F}$ *is correctly defined.*

Proof Let us take Q-bounded Q-based real extrafunctions $F = \text{EF}_Q(f_i)_{i\in\omega}$ and $G = \text{EF}_Q(g_i)_{i\in\omega}$ represented in $\mathbf{F}$. We define $G \cdot F = \text{EF}_Q(g_i \cdot f_i)_{i\in\omega}$. Take another sequence $(h_i)_{i\in\omega}$ that represents F, we show that $G \cdot F = \text{EF}_Q(g_i \cdot h_i)_{i\in\omega}$. Note that all three sequences $(f_i)_{i\in\omega}$, $(h_i)_{i\in\omega}$, and $(g_i)_{i\in\omega}$ belong to $\mathbf{F}^{\omega}$. Then by Lemma

3.2.1, for any seminorm q from Q and the representation $(g_i)_{i\in\omega}$ of G, there is a positive real number c, such that $q(g_i) < c$ for almost all $i = 1, 2, 3, \ldots$. Consequently, we have

$$\lim_{i\to\infty} q(g_i \cdot f_i - g_i \cdot h_i) \leq \lim_{i\to\infty} q(g_i) \cdot q(f_i - h_i) \leq \lim_{i\to\infty} c \cdot q(f_i - h_i) = 0$$

because $\lim_{i\to\infty} q(f_i - h_i) = 0$ and q satisfies Condition N4. As q is a non-negative function, we have

$$\lim_{i\to\infty} q(g_i \cdot f_i - g_i \cdot h_i) = 0$$

As this multiplication is commutative, it is correctly defined for all Q-bounded Q-based real extrafunctions represented in $\mathbf{F}$. Note that the product of Q-bounded real extrafunctions is also Q-bounded.

Proposition is proved.

Proposition 3.2.7 *Any identity that is valid for real functions and involves only operations of addition and/or multiplication of functions and real numbers is also valid for Q-bounded Q-based real extrafunctions represented in* **F**.

Proof is left as an exercise.

In particular, Proposition 3.2.7 gives the following identities for Q-bounded extrafunctions.

1. $$F(x) \cdot G(x) = G(x) \cdot F(x)$$

2. $$F(x) \cdot (G(x) \cdot H(x)) = (F(x) \cdot G(x)) \cdot H(x)$$

3. $$F(x) \cdot (G(x) + H(x)) = F(x) \cdot G(x) + F(x) \cdot H(x)$$

Propositions 3.2.6 and 3.2.7 imply the following result.

Theorem 3.2.5 *If the class* **F** *is closed with respect to multiplication and for each q from Q,* **F** *is a seminormed algebra over* ***R****, then the class* $\mathbf{BF}_{\omega Q}$ *is a linear algebra over* ***R***.

Corollary 3.2.19 *The class* $\mathbf{BF}_{\omega Q}$ *is a module over the linear algebra of all finite/bounded real hypernumbers* $\mathbf{F}\boldsymbol{R}_\omega$

Lemma 3.2.2 *A real function is bounded if and only if it is* Q_{pt}*-bounded.*

Proof is left as an exercise.

Corollary 3.2.20 *The set of all bounded restricted real extrafunctions is a linear algebra.*

Corollary 3.2.21 *The set of all bounded continuously represented restricted real extrafunctions is a linear algebra.*

Corollary 3.2.22 *The set of all bounded compactwise real extrafunctions is a linear algebra.*

3.3 Topological Properties

Taking a set **F** of real functions and a set $Q = \{q_t; t \in K\}$ of seminorms in **F**, we define two types of topology for the set $\mathbf{F}^\omega$ of all sequences of functions from **F**: *local topology* and *global* or *uniform topology*.

Definition 3.3.1 If $\boldsymbol{f} = (f_i)_{i\in\omega}$ is a sequence from $\mathbf{F}^\omega$ and $N = \{r_t; t \in K\}$ is a set of positive real numbers r_t, then a set of the form

$$\mathrm{O}_N\boldsymbol{f} = \{(g_i)_{i\in\omega}; (g_i)_{i\in\omega} \in \mathbf{F}^\omega \ \&\ \forall t \in K \exists k_t \in \boldsymbol{R}^{++} \exists n(t) \in \omega\ \forall i > n(t)(q_t(f_i - g_i) < r_t - k_t)\}$$

is called a locally defined neighborhood of the sequence $\boldsymbol{f} = (f_i)_{i\in\omega}$.

Lemma 3.3.1 *The system of all locally defined neighborhoods determines a topology* $\boldsymbol{\tau}_{lQ}$ *called local Q-topology in the set* $\mathbf{F}^\omega$.

Proof It is necessary to check that the system of locally defined neighborhoods satisfies the neighborhood axioms (cf. Appendix and Kuratowski 1966).

Let us consider an arbitrary sequence $\boldsymbol{f}$ from $\mathbf{F}^\omega$.

NB1: By definition, any neighborhood $\mathrm{O}_N\boldsymbol{f}$ of the sequence $\boldsymbol{f}$ contains $\boldsymbol{f}$.

NB2: Let us consider two neighborhoods $\mathrm{O}_N\boldsymbol{f}$ and $\mathrm{O}_M\boldsymbol{f}$ of the sequence $\boldsymbol{f}$ where $N = \{r_t; t \in K\}$ and $M = \{p_t; t \in K\}$. Let us build the set $L = \{l_t = \min\{r_t, p_t\}; t \in K\}$ and show that the intersection $\mathrm{O}_M\boldsymbol{f} \cap \mathrm{O}_N\boldsymbol{f}$ is equal to the neighborhood $\mathrm{O}_L\boldsymbol{f}$ of $\boldsymbol{f}$. Indeed, if a sequence $\boldsymbol{g}$ belongs both to $\mathrm{O}_N\boldsymbol{f}$ and $\mathrm{O}_M\boldsymbol{f}$, then

$$\forall t \in K \exists k_t \in \boldsymbol{R}^{++} \exists n(t) \in \omega \quad \forall i > n(t)(q_t(f_i - g_i) < r_t - k_t)$$

and

$$\forall t \in K \exists h_t \in \boldsymbol{R}^{++} \exists m(t) \in \omega \quad \forall i > m(t)(q_t(f_i - g_i) < p_t - h_t)$$

Taking some $t \in K$ and defining $u_t = \max\{l_t + k_t - r_t, l_t + h_t - p_t\}$ and $w(t) = \max\{n(t), m(t)\}$, we assume that $r_t < p_t$ and thus, $l_t = r_t$. Then $u_t = \max\{k_t, l_t + h_t - p_t\}$. If $k_t \geq l_t + h_t - p_t$, then $u_t = k_t$ and $q_t(f_i - g_i) < r_t - k_t = l_t - u_t$. If $k_t < l_t + h_t - p_t$, then $u_t = l_t + h_t - p_t$, $p_t = l_t + h_t - u_t$ and $q_t(f_i - g_i) < p_t - h_t = l_t + h_t - u_t - h_t = l_t - u_t$. As the case $r_t > p_t$ is symmetric, we have

$$\forall i > w(t)(q_t(f_i - g_i) < l_t - u_t)$$

Thus, $\boldsymbol{g}$ belongs to $\mathrm{O}_L\boldsymbol{f}$ and consequently, $\mathrm{O}_M\boldsymbol{f} \cap \mathrm{O}_N\boldsymbol{f} \subseteq \mathrm{O}_L\boldsymbol{f}$. At the same time, if $\boldsymbol{g}$ belongs to $\mathrm{O}_L\,\boldsymbol{f}$, then

$$\forall t \in K \exists v_t \in \boldsymbol{R}^{+} \exists w(t) \in \omega \quad \forall i > w(t)(q_t(f_i - g_i) < l_t - v_t)$$

Thus,

$$\forall i > w(t)(q_t(f_i - g_i) < r_t - v_t) \quad \text{and} \quad \forall i > w(t)(q_t(f_i - g_i) < p_t - v_t)$$

It means that $\boldsymbol{g}$ belongs to $\mathrm{O}_M\boldsymbol{f} \cap \mathrm{O}_N\boldsymbol{f}$ and consequently, $O_Mf \cap O_Nf = O_Lf$. Thus, Axiom NB2 is valid.

NB3: Let us consider a neighborhood $\mathrm{O}_N\,\boldsymbol{f}$ of the sequence $\boldsymbol{f}$ and a sequence $\boldsymbol{h} = (h_i)_{i\in\omega}$ that belongs to $\mathrm{O}_N\boldsymbol{f}$. It means that

$$\forall t \in K\, \exists\, k_t \in \boldsymbol{R}^{++} \exists n(t) \quad \forall i > n(t)(q_t(f_i - h_i) < r_t - k_t)$$

Defining $L = \{l_t = (1/3)k_t; t \in K\}$ and $w(t) = \max\{n(t), m(t)\}$ and taking a sequence $\boldsymbol{g}$ that belongs to $\mathrm{O}_L\,\boldsymbol{h}$, we have

$$\forall t \in K\, \exists\, c_t \in \boldsymbol{R}^{+} \exists\, m(t) \quad \forall i > m(t)(q_t(h_i - g_i) < l_t - c_t)$$

At the same time, if $w(t) = \max\{n(t), m(t)\}$, then by the triangle inequality (Axiom N3), for any $i > w(t)$, we have

$$q_t(f_i - g_i) = q_t(f_i - h_i + h_i - g_i) \leq q_t(f_i - h_i) + q_t(h_i - g_i) < r_t - k_t + \frac{1}{3}k_t - c_t$$
$$= r_t - \left(\frac{2}{3}\right)k_t - c_t = r_t - u_t$$

where $u_t = (2/3)\,k_t + c_t$. It means that $\boldsymbol{g}$ belongs to $\mathrm{O}_N\boldsymbol{f}$ and consequently, as $\boldsymbol{g}$ is an arbitrary element from $\mathrm{O}_L\,\boldsymbol{h}$, we have $\mathrm{O}_L\boldsymbol{h} \subseteq \mathrm{O}_N\boldsymbol{f}$. Thus, Axiom NB3 is valid for the system of locally defined neighborhoods.

Lemma is proved.

The topology $\boldsymbol{\tau}_{lQ}$ defines *Q-local convergence* for sequences of functions from **F**. For instance, when **F** is the set of all real functions and $Q = Q_{\mathrm{pt}}$, we have the conventional *pointwise convergence* of sequences of real functions.

Let us assume that the equivalence relation $\approx_Q$ is not trivial in $\mathbf{F}^{\omega}$, i.e., there are, at least, two sequences of functions from **F** that are equivalent with respect to $\approx_Q$.

Proposition 3.3.1 *The topology* $\boldsymbol{\tau}_{lQ}$ *does not satisfy the axiom* $\mathbf{T_0}$ *in* $\mathbf{F}^{\omega}$*, and in particular,* $\mathbf{F}^{\omega}$ *is not a Hausdorff space with respect to this topology.*

Proof Let us consider two arbitrary sequences $\boldsymbol{f} = (f_i)_{i\in\omega}$ and $\boldsymbol{g} = (g_i)_{i\in\omega}$ of real functions from **F** such that $\boldsymbol{f} \approx_Q \boldsymbol{g}$. Then any neighborhood $\mathrm{O}_N\,\boldsymbol{f}$ of the sequence $\boldsymbol{f}$ contains $\boldsymbol{g}$ and any neighborhood $\mathrm{O}_M\,\boldsymbol{g}$ of the sequence $\boldsymbol{g}$ contains $\boldsymbol{f}$.

Proposition is proved (cf. Appendix).

Corollary 3.3.1 *The space* F$(\boldsymbol{R})^{\omega}$ *of all sequences of real functions with the topology of the compact convergence is not a* $\mathbf{T}_0$*-space.*

Definition 3.3.2 If $\boldsymbol{f} = (f_i)_{i\in\omega}$ is a sequence from $\mathbf{F}^{\omega}$ and r is a positive real number, then a set of the form

$\mathrm{O}_r\boldsymbol{f} = \{(g_i)_{i\in\omega}; (g_i)_{i\in\omega} \in F^{\omega}$ and $\exists k \in \boldsymbol{R}^{++}\forall t \in K \exists n(t) \in \omega\ \forall i > n(t)(q_t$ $(f_i - g_i) < r - k)\}$ is called a *uniform neighborhood* of the sequence $\boldsymbol{f} = (f_i)_{i\in\omega}$.

When $N = \{r_t = r \text{ for all } t \in K\}$, Lemma 3.3.1 implies the following result.

Lemma 3.3.2 *The system of all uniform neighborhoods determines a topology* $\boldsymbol{\tau}_{uQ}$ *called* uniform Q-topology *in the set* $\mathbf{F}^{\omega}$.

Proposition 3.3.1 implies the following result.

Corollary 3.3.2 *The uniform topology* $\boldsymbol{\tau}_{uQ}$ *does not satisfy the axiom* $\mathbf{T_0}$ *in* $\mathbf{F}^{\omega}$*, and in particular,* $\mathbf{F}^{\omega}$ *is not a Hausdorff space with respect to this topology.*

The topology $\boldsymbol{\tau}_{uQ}$ defines *Q-uniform convergence* for sequences of functions from **F**. For instance, when **F** is the set of all real functions and $Q = Q_{\mathrm{pt}}$, we have the conventional *uniform convergence* of sequences of real functions.

Definitions imply the following result.

Proposition 3.3.2 *The topology* $\boldsymbol{\tau}_{lQ}$ *is stronger than the topology* $\boldsymbol{\tau}_{uQ}$.

Taking a set **F** of real functions, a set $Q = \{q_t; t \in K\}$ of seminorms in **F** and the set $\mathbf{F}^{\omega}$ of all sequences of functions from **F**, we define two types of topology for the set $\mathbf{E^F}_{\omega Q}$ of Q-based real extrafunctions represented in **F**: local and global or uniform topologies.

Definition 3.3.3 If $F = \mathrm{EF}_Q(f_i)_{i\in\omega}$ is a Q-based real extrafunction represented in **F** and $N = \{r_t; t \in K\}$ is a set of real numbers r_t, then a set of the form

$\mathrm{O}_N F = \{G;$ there is $g = (g_i)_{i\in\omega} \in \mathbf{F}^{\omega}$ such that $G = \mathrm{EF}_Q(g_i)_{i\in\omega}$ and $g = (g_i)_{i\in\omega}$ $\in O_N f$ where $f = (f_i)_{i\in\omega}\}$

is called a *locally defined neighborhood* of the extrafunction F.

Lemma 3.3.3 *The system of all locally defined neighborhoods determines a topology* $\boldsymbol{\delta}_{lQ}$ *called* local Q-topology *in the set* $\mathbf{E}^{\mathbf{F}}_{\omega Q}$.

Proof Let us consider an arbitrary Q-based real extrafunction $F = \mathrm{EF}_Q(f_i)_{i\in\omega}$ represented in **F** and its locally defined neighborhood $\mathrm{O}_N F$ built from the neighborhood $\mathrm{O}_N \boldsymbol{f}$. At first, let us check that the neighborhood $\mathrm{O}_N F$ does not depend on the choice of the representation $(f_i)_{i\in\omega}$ of the extrafunction F. Taking another representation $\boldsymbol{h} = (h_i)_{i\in\omega}$ of the extrafunction F and the locally defined neighborhood $\mathring{\mathrm{O}}_N F$ that is built from the neighborhood $\mathrm{O}_N \boldsymbol{h}$, we show that any extrafunction G from $\mathrm{O}_N F$ also belongs to $\mathring{\mathrm{O}}_N F$.

Indeed, if G belongs to $\mathrm{O}_N\, F$, then there is $(g_i)_{i\in\omega} \in \mathbf{F}^\omega$ such that $G = \mathrm{EF}_Q(g_i)_{i\in\omega}$ and $(g_i)_{i\in\omega} \in \mathrm{O}_N\boldsymbol{f}$ where $\boldsymbol{f} = (f_i)_{i\in\omega}$. It means that

$$\forall t \in K \exists k_t \in \boldsymbol{R}^{++} \exists n(t) \in \omega \quad \forall i > n(t)(q_t(f_i - g_i) < r_t - k_t)$$

As $F = \mathrm{EF}_Q(h_i)_{i\in\omega}$, we have $\lim_{i\to\infty} q_t(f_i - h_i) = 0$ for any $t \in K$, and consequently, for some $l_t < k_t$,

$$\exists m(t) \forall i > m(t)(q_t(f_i - h_i) < l_t)$$

By the properties of seminorms,

$$q_t(h_i - g_i) \le q_t(h_i - f_i) + q_t(f_i - g_i) < r_t - k_t + l_t = r_t - (k_t - l_t) = r_t - p_t$$

where $p_t = k_t - l_t$. Thus, for $c(t) = \max\{n(t), m(t)\}$, we have

$$\forall i > c(t)(q_t(f_i - g_i) < r_t - p_t)$$

As a result, we obtain

$$\forall t \in K \exists p_t \in \boldsymbol{R}^{++} \exists c(t) \in \omega \quad \forall i > c(t)(q_t(h_i - g_i) < r_t - p_t)$$

It means that the extrafunction G belongs to $\mathring{\mathrm{O}}_N\, F$. Thus, we have proved that $\mathrm{O}_N F \subseteq \mathring{\mathrm{O}}_N F$. As the equivalence relation $\approx_Q$ is symmetric, $\mathrm{O}_N F \subseteq \mathring{\mathrm{O}}_N F$ and consequently, $\mathrm{O}_N F = \mathring{\mathrm{O}}_N F$, i.e., the neighborhood $\mathrm{O}_N\, F$ does not depend on the choice of the representation of the extrafunction F.

To prove that the system of all locally defined neighborhoods determines a topology $\boldsymbol{\delta}_{lQ}$ in the set $\mathbf{E}^{\mathbf{F}}_{\omega Q}$, it is necessary to check that the system of locally defined neighborhoods satisfies the neighborhood axioms (cf. Appendix and Kuratowski 1966).

NB1: By definition, any neighborhood $\mathrm{O}_N\, F$ of an extrafunction F contains F.

NB2: Let us consider two neighborhoods $\mathrm{O}_N F$ and $\mathrm{O}_M F$ of the extrafunction F where $N = \{r_t; t \in K\}$ and $M = \{p_t; t \in K\}$.

If $\pi_Q : \mathbf{F}^\omega \to \mathbf{E}^{\mathrm{F}}{}_{\omega Q}$ is the natural projection, i.e., $\pi_Q(\boldsymbol{f} = (f_i)_{i\in\omega}) = EF_Q(f_i)_{i\in\omega}$ then the neighborhood $\mathrm{O}_N F$ is equal to the projection π_Q $(\mathrm{O}_N\boldsymbol{f})$ and the neighborhood $\mathrm{O}_M F$ is equal to the projection π_Q $(\mathrm{O}_M\, \boldsymbol{f})$. By Lemma 3.3.1, there is a neighborhood $\mathrm{O}_L\boldsymbol{f}$ that belongs to the intersection of both neighborhoods $\mathrm{O}_N\boldsymbol{f}$ and $\mathrm{O}_M\boldsymbol{f}$. Then its projection π_Q $(\mathrm{O}_L\boldsymbol{f})$ is a neighborhood of F that belongs to the intersection of both neighborhoods $\mathrm{O}_N F = \pi_Q(O_N\boldsymbol{f})$ and $\mathrm{O}_M F = \pi_Q \mathrm{O}_M\boldsymbol{f}$. Thus, Axiom NB2 is valid.

NB3: Let us consider a neighborhood $\mathrm{O}_N F$ of the extrafunction F and an extrafunction G that belongs to $\mathrm{O}_N F$. It means that there is a sequence

$\boldsymbol{g} = (g_i)_{i\in\omega} \in \mathbf{F}^{\omega}$ such that $G = \mathrm{EF}_Q(g_i)_{i\in\omega}$ and $(g_i)_{i\in\omega} \in \mathrm{O}_N\boldsymbol{f}$ where $\boldsymbol{f} = (f_i)_{i\in\omega}$. By Lemma 3.3.1, there is a neighborhood $\mathrm{O}_L\,\boldsymbol{g}$ such that

$\mathrm{O}_L\boldsymbol{g} \subseteq \mathrm{O}_N\boldsymbol{f}$. Then its projection $\pi_Q(\mathrm{O}_L\boldsymbol{g})$ is a neighborhood of G and $\mathrm{O}_LG = \pi_Q(\mathrm{O}_L\boldsymbol{g}) \subseteq \pi_Q(\mathrm{O}_N\boldsymbol{f}) = O_NF$. Thus, Axiom NB3 is also valid.

Lemma is proved.

Theorem 3.3.1 *The topology* $\boldsymbol{\delta}_{lQ}$ *satisfies the axiom* $\mathbf{T}_2$*, and thus,* $\mathbf{E}^{\mathbf{F}}_{\omega Q}$ *is a Hausdorff space with respect to this topology.*

Proof Let us consider two arbitrary extrafunctions $F = \mathrm{EF}_Q(f_i)_{i\in\omega}$ and $G = \mathrm{EF}_Q(g_i)_{i\in\omega}$ from $\mathbf{E}^{\mathbf{F}}_{\omega Q}$. If $F \neq G$ in $\mathbf{E}^{\mathbf{F}}_{\omega Q}$, then any sequences $\boldsymbol{f} = (f_i)_{i\in\omega} \in F$ and $\boldsymbol{g} = (g_i)_{i\in\omega} \in G$ satisfy the following condition: there is a seminorm q from Q and a positive number k such that for any $n \in \omega$, there is an $i > n$ such that $q(f_i - g_i) > k$. This condition makes it possible to choose an infinite set M of natural numbers such that for any $m \in M$ the inequality $q(f_m - g_m) > k$ is valid.

Let us take $l = k/4$ and consider two uniform neighborhoods $\mathrm{O}_l\boldsymbol{f}$ and $\mathrm{O}_l\boldsymbol{g}$ of the sequences $\boldsymbol{f}$ and $\boldsymbol{g}$ in $\mathbf{F}^{\omega}$. If $\mathbf{F}^{\omega} \to \mathbf{E}^{\mathbf{F}}_{\omega Q}$ is the natural projection, then the projections $\pi_Q(\mathrm{O}_l\boldsymbol{f})$ and $\pi_Q(\mathrm{O}_l\boldsymbol{g})$ of these neighborhoods $\mathrm{O}_l\boldsymbol{f}$ and $\mathrm{O}_l\boldsymbol{g}$ are neighborhoods O_lF and O_lG of the extrafunctions F and G with respect to the topology $\boldsymbol{\delta}_{lQ}$. By construction, $\pi_Q(\mathrm{O}_l\boldsymbol{f}) \cap \pi_Q(\mathrm{O}_l\boldsymbol{g}) = \varnothing$. To prove this, we suppose that this is not true. Then there is a an extrafunction $H = \mathrm{EF}_Q(h_i)_{i\in\omega}$ from $\mathbf{E}^{\mathbf{F}}_{\omega Q}$ that is an element of the set $\pi_Q(\mathrm{O}_l\boldsymbol{f}) \cap \pi_Q(\mathrm{O}_l\boldsymbol{g})$. This implies that there are sequences $\boldsymbol{u} = (u_i)_{i\in\omega}$ and $\boldsymbol{v} = (v_i)_{i\in\omega}$ from $\mathbf{F}^{\omega}$ for which $\pi_Q(\boldsymbol{u}) = \pi_Q(\boldsymbol{v}) = H$, $\boldsymbol{u} \in \mathrm{O}_l\boldsymbol{f}$ and $\boldsymbol{v} \in \mathrm{O}_l\mathbf{g}$.

The equality $\pi_Q(\boldsymbol{u}) = \pi_Q(\boldsymbol{v})$ implies that for the chosen seminorm q and number k, the following condition (*) is valid: $\exists m \in \omega \forall i{>}m(q(u_i - v_i){<}k/3)$. The set M, which is determined above, is infinite. So, there is $j \in M$ such that it is greater than m and $q(u_j - v_j)^3 q(f_j - g_j) - q(f_j - u_j) - q(g_j - v_j)^3$ $k - k/4 - k/4 = k/2{>}k/3$ because $f_j - g_j = f_j - u_j + u_j - v_j - g_j + v_j$ and by properties of seminorms, $q(f_j - g_j)q(f_j - u_j) + q(u_j - v_j) + q(g_j - v_j)$, $q(f_j - u_j){<}l - r{<}l = k/4$, $q(g_j - v_j){<}l - p{<}l = k/4$, and $q(v_j - g_j) = q(g_j - v_j)$. This contradicts the condition (*), according to which $q(u_j - v_j){<}k/3$.

Consequently, the assumption is not true, and $\pi_Q(\mathrm{O}_l\boldsymbol{f}) \cap \pi_Q(\mathrm{O}_l\boldsymbol{g}) = \mathrm{O}_lF \cap \mathrm{O}_lG = \varnothing$. Theorem is proved because $F = \mathrm{EF}_Q(f_i)_{i\in\omega}$ and $G = \mathrm{EF}_Q(g_i)_{i\in\omega}$ are arbitrary extrafunctions from $\mathbf{E}^{\mathbf{F}}_{\omega Q}$.

The topology $\boldsymbol{\delta}_{lQ}$ defines *Q-local convergence* of Q-based real extrafunctions represented in $\mathbf{F}$.

Definition 3.3.4 If $F = \mathrm{EF}_Q(f_i)_{i\in\omega}$ is a Q-based real extrafunction represented in $\mathbf{F}$ and r is a real number, then a set of the form

$$\mathrm{O}_rF = \{G;\text{there is } g = (g_i)_{i\in\omega} \in \mathbf{F}^{\omega}\text{such that } G = \mathrm{EF}_Q(g_i)_{i\in\omega} \text{ and } g = (g_i)_{i\in\omega} \in \mathrm{O}_rf \text{ where } f = (f_i)_{i\in\omega}\}$$

is called a *uniform neighborhood* of the extrafunction F.

When $N = \{r_t = r \text{ for all } t \in K\}$, Lemma 3.3.3 implies the following result.

Lemma 3.3.4 *The system of all uniform neighborhoods determines a topology* $\boldsymbol{\delta}_{uQ}$ *called uniform Q-topology in the set* $\mathbf{E}^{\mathbf{F}}_{\omega Q}$.

Theorem 3.3.1 implies the following result.

Corollary 3.3.4 *The uniform topology* $\boldsymbol{\delta}_{uQ}$ *satisfies the axiom* $\mathbf{T}_2$*, and thus,* $\mathbf{E}^{\mathbf{F}}_{\omega Q}$ *is a Hausdorff space with respect to this topology.*

The topology $\boldsymbol{\delta}_{uQ}$ defines *Q-uniform convergence* of *Q*-based real extrafunctions represented in **F**.

Corollary 3.3.5 *The space of all restricted real extrafunctions with the uniform topology is a Hausdorff space.*

Corollary 3.3.6 *The space of all continuously represented restricted real extrafunctions with the uniform topology is a Hausdorff space.*

Corollary 3.3.7 *The space of all real compactwise extrafunctions with the uniform topology is a Hausdorff space.*

Corollary 3.3.8 *The space of all real extended distributions with the uniform topology is a Hausdorff space.*

Definitions imply the following result.

Proposition 3.3.3 *The topology* $\boldsymbol{\delta}_{lQ}$ *is stronger than the topology* $\boldsymbol{\delta}_{uQ}$.

Chapter 4
How to Differentiate Any Real Function

Progress is measured by the degree of differentiation within a society.

Herbert Read (1893–1968)

Here we explore what advantages hypernumbers and extrafunctions offer for differentiation of real functions. In Sect. 4.1, basic elements of the theory of approximations are presented. We consider approximations of two types: approximations of a point by pairs of points, which are called A-approximations and used for differentiation, and approximations of topological spaces by their subspaces, which are called B-approximations and used for integration.

In Sect. 4.2, basic elements of the *theory of hyperdifferentiation*, also called the *extended differential calculus*, are presented. Here we consider hyperdifferentiation only for real functions, defining their *sequential partial derivatives*. Various properties of sequential partial derivatives are obtained. Some of these properties are similar to properties of conventional derivatives. For instance, the sequential partial derivative of the sum of two functions is equal to the sum of sequential partial derivatives of each of these functions (Theorem 4.2.2). Other properties of sequential partial derivatives are essentially different from properties of conventional derivatives. For instance, the sequential partial derivative of any total real function always exists. However, as a rule, it is not unique, and if we have positive infinite sequential partial derivative of a real function f at some point, then for any finite or infinite positive hypernumber α, there is a sequential partial derivative of the function f at the same point such that its value at the same point is larger than α (Theorem 4.2.1).

Hyperdifferentiation and differentiation of extrafunctions is studied by Burgin (1993, 2002) and hyperdifferentiation and differentiation of complex functions is studied by Burgin and Ralston (2004) and Burgin (2010).

M. Burgin, *Hypernumbers and Extrafunctions: Extending the Classical Calculus*, SpringerBriefs in Mathematics, DOI 10.1007/978-1-4419-9875-0_4,

4.1 Approximations

Here we consider two types of approximations: approximations of topological spaces called B-approximations and approximations of points in real vector spaces (with the emphasis on the one-dimensional spaces) called A-approximations. B-approximations are used for integration, while A-approximations are used for differentiation.

Let X be a topological space.

Definition 4.1.1 A system $A = \{X_i; i \in I\}$ of subspaces X_i of the space X is called a *lower (topological) approximation* or simply, B-*approximation* of X if X is equal to the closure $\mathrm{Cl}\left(\bigcup_{i\in I} X_i\right)$ of the union $\bigcup_{i\in I} X_i$.

Example 4.1.1 The system $A = \{(n, n+1); n \in \boldsymbol{Z}\}$ of open intervals $(n, n + 1)$ is a topological approximation of the real line $\boldsymbol{R}$.

Any partition of X also is a topological approximation of the same space.

Lemma 4.1.1 *If a system $A = \{X_i; i \in I\}$ of subspaces X_i of the space X contains a subsystem that is a* B-*approximation of X, then A is a* B-*approximation of the space X.*

Indeed, if $C = \{X_j; j \in J \subseteq I\}$ is a B-approximation of the space X, then $X = \mathrm{Cl}\left(\bigcup_{i\in J} X_i\right)$. Consequently, $X = \mathrm{Cl}\left(\bigcup_{i\in I} X_i\right)$ as all X_i are subspaces of the space X. Thus, by Definition 4.1.1, A is a B-approximation of the space X.

Let X and Y be topological spaces. As the union of two closed sets is a closed set, we have $\mathrm{Cl}(U \bigcup V) = \mathrm{Cl}(U) \bigcup Cl(V)$ for any sets U and V. This gives us the following result.

Proposition 4.1.1 *If $A = \{X_i; i \in I\}$ is a B-approximation of X and $B = \{Y_j; j \in J\}$ is a B-approximation of Y*, then *$A \bigcup B$ is a B-approximation of $X \bigcup Y$.*

Proof is left as an exercise.

B-approximations are used to define integrals in infinite dimensional spaces, e.g., the path integral (cf. Burgin 1995, 2008/2009).

Let L be a vector space and $a \in L$.

Definition 4.1.2 A sequence $R = \{<a_i, b_i>; i \in \omega\}$ of pairs of collinear elements from L with $b_i = k_i a_i$ where a_i is a real number larger than 1 is called an *upper approximation* or A-*approximation* of the point a if $a = \lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i$.

Here we study the one-dimensional case of A-approximations, i.e., when $L = \boldsymbol{R}$. More general A-approximations are used for directional dimensional partial derivatives of functions, which are not considered in this book.

Let $a \in \boldsymbol{R}$.

Definition 4.1.3 A sequence $R = \{<a_i, b_i>; i \in \omega\}$ of pairs of real numbers with $a_i < b_i$ is called an *upper approximation* or A-*approximation* of the real number (point in the space $\boldsymbol{R}$) a if $a = \lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i$.

As we know from the calculus, the limit of a subsequence of a converging sequence coincides with the limit of this sequence (cf. Gemignani 1971; Ross 1996; Burgin 2008a), we have the following result.

Lemma 4.1.2 *Any infinite subsequence of an* A*-approximation of a point a is an* A*-approximation of the point a.*

As we know from the calculus (cf. Gemignani 1971; Ross 1996; Edwards and Penney 2002; Burgin 2008a), if two converging sequences l and h have the same limit a, then a is the limit of the sequence that is the union of the sequences l and h. This gives us the following result.

Lemma 4.1.3 *The set-theoretical union of two* A*-approximations of a point a is an* A*-approximation of the point a.*

There are different useful types of A-approximations.

Definition 4.1.4 An A-approximation $R = \{<a_i, b_i>; i \in \omega\}$ of a is called:

(a) *Two-sided* if $a_i < a < b_i$ for all $i \in \omega$;
(b) *Left* if $a_i < b_i \leq a$ for all $i \in \omega$;
(c) *Right* if $a \leq a_i < b_i$ for all $i \in \omega$;
(d) *Centered* if $a_i = a - h_i$ and $b_i = a + h_i$ for all $i \in \omega$;

Definition 4.1.5 A right (left) A-approximation $R = \{<a_i, b_i>; i \in \omega\}$ of a point a is called:

(a) *Stable* if there is $n \in \omega$ such that $a_i = a$ ($b_i = a$) when $i > n$
(b) *Almost stable* if $\lim_{i\to\infty}(a_i - a)/(b_i - a_i) = 0$ ($\lim_{i\to\infty}(a - b_i)/(b_i - a_i) = 0$)
(c) *Monotone* if $a_{i+1} \leq a_i < b_{i+1} \leq b_i$ ($a_i \leq a_{i+1} < b_i \leq b_{i+1}$) for all $i \in \omega$

Definition 4.1.6 A two-sided A-approximation $R = \{<a_i, b_i>; i \in \omega\}$ of a point a is called *monotone* if $a_i \leq a_{i+1} < b_{i+1} \leq b_i$ for all $i \in \omega$.

Stable A-approximations are used in the calculus to define conventional derivatives (cf. Ross 1996; Burgin 2008a).

We use the following denotations:

appr(a) is the set of all A-approximations of a point a
bappr(a) is the set of all two-sided A-approximations of a point a
lappr(a) is the set of all left A-approximations of a point a
rappr(a) is the set of all right A-approximations of a point a
cappr(a) is the set of all centered A-approximations of a point a
srappr(a) is the set of all stable right A-approximations of a point a
slappr(a) is the set of all stable left A-approximations of a point a
asrappr(a) is the set of all almost stable right A-approximations of a point a
aslappr(a) is the set of all almost stable left A-approximations of a point a

Lemma 4.1.4 $appr(a) = bappr(a) \bigcup lappr(a) \bigcup rappr(a)$.

Proof is left as an exercise.

Lemma 4.1.5 *Any stable* A-*approximation is almost stable, i.e., $srappr(a) \subseteq asrappr(a)$ and $slappr(a) \subseteq aslappr(a)$.*

Proof is left as an exercise.

Let $Q, R \in appr(a)$.

Definition 4.1.7 An A-approximation $Q = \{<c_i, d_i>; i \in \omega\}$ of a point a is called a *subapproximation* of an A-approximation $R = \{<a_i, b_i>; i \in \omega\}$ of the point a if there is an injection $f: \omega \to \omega$ such that $c_i = a_{f(i)}$ and $d_i = b_{f(i)}$ for all $i \in \omega$. It is denoted by $Q \subseteq R$.

Lemma 4.1.6 *If $Q \subseteq R$ and $R \in bappr(a)$ $(R \in lappr(a), R \in rappr(a))$, then $Q \in bappr(a)$ $(Q \in lappr(a), Q \in rappr(a))$, respectively.*

Proof is left as an exercise.

Lemma 4.1.7 *If $Q \subseteq R$ and $R \subseteq T$, then $Q \subseteq T$.*

Proof is left as an exercise.

Lemma 4.1.8 *If $Q \subseteq R$ and R is a monotone* A-*approximation of a point a, then Q is also a monotone A-approximation of a.*

Definition 4.1.8 A *mesh* m_R of an A-approximation $R = \{<a_i, b_i>; i \in \omega\}$ is the function $m_R : \mathbf{N} \to \mathbf{R}$ such that $m_R(n) = |b_n - a_n|$ for all $n \in \mathbf{N}$.

Lemma 4.1.9 *For any* A-*approximation R of a point a, $m_R(n) \to 0$ when $n \to \infty$.*

Indeed, $\lim_{i\to\infty}|b_n - a_n| = 0$ because $\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = a$.

Lemma 4.1.10 *If R* is a monotone two-sided A-*approximation of a point a, then m_R is a decreasing function.*

Indeed, each pair $<a_i, b_i>$ is inside the pair $<a_{i-1}, b_{i-1}>$.

Definition 4.1.9 An A-approximation $Q = \{<c_i, d_i>; i \in \omega\}$ is *dominated* by an A-approximation $R = \{<a_i, b_i>; i \in \omega\}$ if $[c_i, d_i] \subseteq [a_i, b_i]$ for all $i \in \omega$. It is denoted by $Q \preccurlyeq R$.

Lemma 4.1.11 *Relation $\preccurlyeq$ is a partial order in the set of all* A-*approximations.*

Proof is left as an exercise.

Lemma 4.1.12 *If $Q \preccurlyeq R$, then $m_Q(n) \leq m_R(n)$ as functions.*

Indeed, $R = \{<a_i, b_i>; i \in \omega\}$ and $Q = \{<c_i, d_i>; i \in \omega\}$ are A-approximation of a point a and $Q \preccurlyeq R$, then $[c_i, d_i] \subseteq [a_i, b_i]$ implies $|c_i, d_i| \leq |a_i, b_i|$ and thus, $m_Q(n) \leq m_R(n)$ for all $n = 1, 2, 3, \ldots$.

Lemma 4.1.13 *If R* is a monotone two-sided A-*approximation, then $Q \subseteq R$ implies $Q \preccurlyeq R$.*

Indeed, if $R = \{<a_i, b_i>; i \in \omega\}$ is a monotone two-sided A-approximation of a point a, then $[a_{i+k}, b_{i+k}] \subseteq [a_i, b_i]$ for all $i, k \in \omega$.

Definition 4.1.10 An A-approximation $R = \{<a_i, b_i>; i \in \omega\}$ of a point *a belongs to* a neighborhood (*c*, *d*) of *a* if all points a_i, b_i belong to the interval (*c*, *d*). It is denoted by $R \subseteq (c, d)$.

Lemma 4.1.14 *For any neighborhood (c, d) of a point a*, each A-*approximation R of a has a subapproximation Q that belongs to (c, d).*

Indeed, as $\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = a$, there is a natural number *n* such that $a_i, b_i \in (c, d)$ when $i > n$.

Lemma 4.1.15 *If* $R \subseteq (c, d)$ *and Q is a subapproximation of R, then* $Q \subseteq (c, d)$.

Indeed, all pairs of points from *Q* also belong to *P*.

Lemma 4.1.16 *If* $Q \leqslant R$ *and* $R \subseteq (c, d)$, *then* $Q \subseteq (c, d)$.

Indeed, all pairs of points from *Q* are inside corresponding pairs of points from *P* and thus, they belong to the same interval.

A-approximations are used for hyperdifferentiation of real functions, which is studied in the next section.

4.2 Hyperdifferentiation

Let us assume that $X, Y \subseteq \boldsymbol{R}$, $f : X \to Y$ is a function, $a \in X$, $b \in \boldsymbol{R}$, X contains some open interval $(a - k, a + k)$, and $r \in \boldsymbol{R}^+$. Taking an A-approximation $R = \{<a_i, b_i>; i \in \omega\}$ of a point (i.e., real number) *a*, such that $R \subseteq (a - k, a + k)$, we define the sequential partial derivative of *f* at *a* with respect to *R*.

Definition 4.2.1 A hypernumber β is called the *sequential partial derivative* of the function *f*(*x*) at the point *a* with respect to the A-approximation *R* if $\beta = \mathrm{Hn}(c_i)_{i\in\omega}$ where $c_i = (f(b_i) - f(a_i))/(b_i - a_i) = \Delta_i^R f/\Delta_i^R x$, $\Delta_i^R f = f(b_i) - f(a_i)$ and $\Delta_i^R x = b_i - a_i$ for all $i = 1, 2, 3, \ldots$.

It is denoted by

$$\beta = \partial/_{\partial R} f(a)$$

We call these derivatives sequential to emphasize their distinction from the conventional partial derivatives $\partial/_{\partial x} f$, $\partial/_{\partial y} f$, and $\partial/_{\partial z} f$, which are naturally called *dimensional partial derivatives*. Choosing relevant approximations of points in a (three-dimensional) Euclidean space, it is possible to express dimensional partial derivatives by sequential partial derivatives.

We see that in contrast to conventional derivatives, the sequential partial derivative of any total function *f*(*x*) at any point *a* of $\boldsymbol{R}$ is always defined.

Example 4.2.1 Let us take the function $f(x) = |x|$. Its sequential partial derivative $\partial/_{\partial R} f(0)$ at 0 may be equal to any hypernumber from the interval [−1, 1], which include all numbers from this interval, as well as proper hypernumbers. For instance, the value of the derivative $\partial/_{\partial R} f(0)$ may be equal to different oscillating hypernumbers, e.g., to the hypernumber $\mathrm{Hn}(a_i)_{i\in\omega}$ with $a_{2k} = 1$, $a_{2k-1} = -1$, and $k = 1, 2, 3, \ldots$.

Definition 4.2.1 shows that if f is a total function in an interval (a, b) (on the whole real line $\boldsymbol{R}$), the sequential partial derivative of f with respect to any A-approximation R is defined for any point of (a, b) (of $\boldsymbol{R}$). In particular, this shows that now we can differentiate any total real function. Even more, in some cases, it is possible and reasonable to differentiate a partial real function even at points where it is not defined. A sequential partial derivative $\partial/_{\partial R} f(a)$ can exist even for a discontinuous function f. It is only necessary that f is defined at all points of the A-approximation R.

The procedure of taking a sequential partial derivative is called *hyperdifferentiation.*

Taking an A-approximation R, we see that in a general case, the sequential partial derivative $\partial/_{\partial R} f$ of a real function f with respect to R is a restricted extrafunction when we take $\partial/_{\partial R} f(x)$ for all real numbers x.

Let us find some properties of sequential partial derivatives, which are simply called partial derivatives in what follows. It does not cause any inconvenience because we do not consider here dimensional partial derivatives.

Lemma 4.2.1 *If R is an* A*-approximation of a point a and $\partial/_{\partial R} f(a)$ is a real number, then $\partial/_{\partial Q} f(a) = \partial/_{\partial R} f(a)$ for any subapproximation Q of R.*

Indeed, if a sequence has a limit, then it is the limit of any of its subsequence.

Lemmas 4.1.15 and 4.2.1 implies the following result.

Proposition 4.2.1 *For any neighborhood (c, d) of a point a and any A-approximation R of a for which $\partial/_{\partial R} f(a)$ is a real number, there is a subapproximation Q such that Q belongs to (c, d) and $\partial/_{\partial Q} f(a) = \partial/_{\partial R} f(a)$.*

The following result demonstrates that Lemma 4.2.1 is not true when $\partial/_{\partial R} f(a)$ is equal to an arbitrary hypernumber.

Theorem 4.2.1

(a) *If $\alpha = \partial/_{\partial R} f(a)$ and α is either an infinitely increasing or infinite positive or infinite expanding hypernumber, then for any hypernumber $\beta \in \boldsymbol{R}_\omega$, there is a subapproximation Q* of *R such that $\partial/_{\partial Q} f(a) > \beta$.*

(b) *If $\alpha = \partial/_{\partial R} f(a)$ and α is either an infinitely decreasing or infinite negative or infinite expanding hypernumber, then for any hypernumber $\eta \in \boldsymbol{R}_\omega$, there is a subapproximation P* of *R such that $\partial/_{\partial P} f(a) < \eta$.*

Proof Let us assume that $\partial/_{\partial R} f(a) = \alpha = \mathrm{Hn}(c_i)_{i \in \omega}$ is an infinitely increasing hypernumber where $R = \{<a_i, b_i>; i \in \omega\}$ and $c_i = (f(b_i) - f(a_i))/(b_i - a_i) = \Delta_i{}^R f/\Delta_i{}^R x$ and $\beta = \mathrm{Hn}(b_i)_{i \in \omega}$ is an arbitrary hypernumber. In this case, there is a number $c_j = c_{j(1)}$ that is larger than b_1. Then for each $i \in \omega$, we can find a number $c_j = c_{j(i)}$ that is larger than b_i. Taking $d_i = c_{j(i)}$, we obtain a real hypernumber $\gamma = \mathrm{Hn}(d_i)_{i \in \omega}$, which is larger than β by Definition 2.2.2. Because each number d_i is equal to some $(f(b_j) - f(a_j))/(b_j - a_j)$, it is possible to find a subapproximation Q of R such that $\gamma = \partial/_{\partial Q} f(a) > \beta$.

When α is an infinite positive or infinite expanding hypernumber, similar constructions give us the necessary real hypernumber γ larger than β.

Part (b) of Theorem 4.2.1 is treated in a similar way.

Theorem is proved.

Corollary 4.2.1 *If* $\alpha = \partial/_{\partial R} f(a)$ *and Spec* α *is not bounded from above (from below), then for any hypernumber* $\beta \in \boldsymbol{R}_\omega$, there is a subapproximation *Q of R such that* $\partial/_{\partial Q} f(a) > \beta$ *(correspondingly,* $\partial/_{\partial Q} f(a) < \beta$).

Indeed, when *Spec* α is not bounded from above, then α is either an infinitely increasing hypernumber or infinite positive hypernumber or infinite expanding hypernumber, and the statement directly follows from Theorem 4.2.1(a).

When *Spec* α is not bounded from below, then α is either an infinitely decreasing hypernumber or infinite negative hypernumber, or infinite expanding hypernumber and the statement directly follows from Theorem 4.2.1(b).

Corollary 4.2.2 *If* $\alpha = \partial/_{\partial R} f(a)$ *and* α *is an infinite expanding hypernumber, then for any hypernumber* $\beta \in \boldsymbol{R}_\omega$*, there is a subapproximation Q* of *R such that* $\partial/_{\partial Q} f(a) > \beta$ *and a subapproximation P* of *R such that* $\partial/_{\partial P} f(a) < \beta$.

Theorem 4.2.2 *For any real functions f* and g and any A-*approximation* $R = \{<a_i, b_i>; i \in \omega\}$ *of a point (i.e., real number) a, we have*

$$\partial/_{\partial R} (f + g)(a) = \partial/_{\partial R} f(a) + \partial/_{\partial R}\, g(a)$$

Proof By definition, $\partial/_{\partial R} (f + g)(a) = \mathrm{Hn}(c_i)_{i\in\omega}$ where $c_i = ((f(b_i) + g(b_i)) - (f(a_i) + g(a_i)))/(b_i - a_i) = \Delta_i^R(f + g)/\Delta_i^R x$ with $\Delta_i^R f = f(b_i) - f(a_i)$ and $\Delta_i^R x = b_i - a_i$ for all $i = 1, 2, 3, \ldots$.

Let us consider one element c_i from the sequence $(c_i)_{i\in\omega}$ that defines $\partial/_{\partial R} (f + g)(a)$.

$$\begin{aligned}\Delta_i^R(f + g)/\Delta_i^R x &= \frac{(f(b_i) + g(b_i)) - (f(a_i) + g(a_i))}{b_i - a_i}\\ &= \frac{(f(b_i) - f(a_i)) + (g(b_i) - g(a_i))}{b_i - a_i}\\ &= \frac{g(b_i) - g(a_i)}{b_i - a_i} + \frac{f(b_i) - f(a_i)}{b_i - a_i}\end{aligned}$$

As this is true for all $i \in \omega$, by Theorem 3.2.1, we have

$$\begin{aligned}\partial/_{\partial R} (f + g)(a) &= \mathrm{Hn}(\Delta_i^R(f + g)/\Delta_i^R x)_{i\in\omega}\\ &= \mathrm{Hn}(\Delta_i^R(f)/\Delta_i^R x)_{i\in\omega} + \mathrm{Hn}(\Delta_i^R(g)/\Delta_i^R x)_{i\in\omega}\\ &= \partial/_{\partial R} f(a) + \partial/_{\partial R}\, g(a)\end{aligned}$$

Theorem is proved.

Theorem 4.2.3 *For any real function f,* any real number c and any A-*approximation* $R = \{<a_i, b_i>; i \in \omega\}$ *of a point (i.e., real number) a, we have*

$$\partial/_{\partial R} (c \cdot f(a)) = c \cdot (\partial/_{\partial R} f(a))$$

Proof By definition, $\partial/_{\partial R}(cf)(a) = \text{Hn}(d_i)_{i\in\omega}$ where $d_i = (cf(b_i) - cf(a_i))/(b_i - a_i) = \Delta_i{}^R(cf)/\Delta_i{}^R x$, $\Delta_i{}^R f = f(b_i) - f(a_i)$ and $\Delta_i{}^R x = b_i - a_i$ for all $i = 1, 2, 3, \ldots$. Then by Theorem 3.2.1, we have

$$\partial/_{\partial R}(cf)(a) = \text{Hn}(\Delta_i{}^R(cf)/\Delta_i{}^R x)_{i\in\omega} = \text{Hn}(c\Delta_i{}^R(f)/\Delta_i{}^R x)_{i\in\omega}$$
$$= c\text{Hn}(\Delta_i{}^R(f)/\Delta_i{}^R x)_{i\in\omega} = c \cdot \partial/_{\partial R}(cf)(a)$$

Theorem is proved.

Corollary 4.2.3 *For any real functions f and g, any real numbers c and d and any* A-*approximation* $R = \{<a_i, b_i>; i \in \omega\}$ *of a point (i.e., real number) a, we have*

$$\partial/_{\partial R}(c \cdot f + d \cdot g)(a) = c \cdot (\partial/_{\partial R} f(a)) + d \cdot (\partial/_{\partial R} g(a))$$

It means that for any system $\boldsymbol{Q}$ of A-approximations in which there is a one-to-one correspondence between all points from $\boldsymbol{R}$ and all approximations from $\boldsymbol{Q}$, i.e., $\boldsymbol{Q} = \{R_x; x \in \boldsymbol{R}\}$, the sequential partial derivative $\partial/_{\partial R}$ defines a linear operator from the space of all (total) real functions into the space of all restricted real extrafunctions. In general, hyperdifferentiation defines a multivalued operator.

An important particular case of Corollary 4.2.3 is the following result.

Corollary 4.2.4 *For any real functions f and g and any* A-*approximation* $R = \{<a_i, b_i>; i \in \omega\}$ *of a point (i.e., real number) a, we have*

$$\partial/_{\partial R}(f - g)(a) = \partial/_{\partial R} f(a) - \partial/_{\partial R} g(a)$$

Theorem 4.2.4 *For any continuous at a point (real number) a functions f and g and any* A-*approximation* $R = \{<a_i, b_i>; i \in \omega\}$ *of a*, we have

$$\partial/_{\partial R}(f \cdot g)(a) = g(a) \cdot \partial/_{\partial R} f(a) + f(a) \cdot \partial/_{\partial R} g(a)$$

Proof By definition, $\partial/_{\partial R}(f \cdot g)(a) = \text{Hn}(c_n)_{n\in\omega}$ where $c_n = (f(b_n)g(b_n) - f(a_n)g(a_n))/(b_n - a_n) = \Delta_n{}^R(fg)/\Delta_n{}^R x$, $\Delta_n{}^R f = f(b_n) - f(a_n)$ and $\Delta_n{}^R x = b_n - a_n$ for all $n = 1, 2, 3, \ldots$. Let us consider one of these ratios.

$$\Delta_n{}^R(fg)/\Delta_n{}^R x = \frac{f(b_n)g(b_n) - f(a_n)g(a_n)}{b_n - a_n} = \frac{f(b_n)g(b_n) - f(b_n)g(a_n) + f(b_n)g(a_n) - f(a_n)g(a_n)}{b_n - a_n}$$
$$= \frac{f(b_n)g(b_n) - f(b_n)g(a_n)}{b_n - a_n} + \frac{f(b_n)g(a_n) - f(a_n)g(a_n)}{b_n - a_n}$$
$$= f(b_n)\frac{g(b_n) - g(a_n)}{b_n - a_n} + g(a_n)\frac{f(b_n) - f(a_n)}{b_n - a_n}$$
$$= f(b_n)(\Delta_n{}^R(g)/\Delta_n{}^R x) + g(a_n)(\Delta_n{}^R(f)/\Delta_n{}^R x)$$

As this is true for all $n \in \omega$, we have

$$\mathrm{Hn}(c_n)_{n\in\omega} = \mathrm{Hn}(d_n)_{n\in\omega} \quad \text{where } d_n = f(b_n)(\Delta_n{}^R(g)/\Delta_n{}^R x) + g(a_n)\Delta_n{}^R(f)/\Delta_n{}^R x$$

As f is continuous at the point a and $\lim_{n\to\infty} b_n = a$, we have $\mathrm{Hn}(f(b_n))_{n\in\omega} = f(a)$.

As g is continuous at the point a and $\lim_{n\to\infty} a_n = a$, we have $\mathrm{Hn}(g(a_n))_{n\in\omega} = g(a)$.

Continuity of f and g at the point a implies that both functions are bounded in the neighborhood of a (cf. Burgin 2008a). Consequently, by Theorems 2.2.1 and 3.2.3, we have

$$\begin{aligned} \partial/_{\partial R}(f\cdot g)(a) &= \mathrm{Hn}(c_n)_{n\in\omega} = \mathrm{Hn}(d_n)_{n\in\omega} \\ &= \mathrm{Hn}(f(b_n))_{n\in\omega}\cdot \mathrm{Hn}(\Delta_n{}^R(g)/\Delta_n{}^R x)_{n\in\omega} + \mathrm{Hn}(g(a_n))_{n\in\omega} \\ &\quad \cdot \mathrm{Hn}(\Delta_n{}^R(f)/\Delta_n{}^R x)_{n\in\omega} \\ &= f(a)\cdot \mathrm{Hn}(\Delta_n{}^R(g)/\Delta_n{}^R x)_{n\in\omega} + g(a)\cdot \mathrm{Hn}(\Delta_n{}^R(f)/\Delta_n{}^R x)_{n\in\omega} \\ &= g(a)\cdot \partial/_{\partial R}f(a) + f(a)\cdot \partial/_{\partial R}g(a) \end{aligned}$$

Theorem is proved.

To understand how weak the condition that both functions f and g are continuous at the point a is in comparison with the standard situation, we remind that any differentiable (in the classical sense) function is automatically continuous, and even a fuzzy differentiable function is continuous (Burgin 2008a).

Remark 4.2.1 The condition that both functions f and g are continuous at the point a is essential in Theorem 4.2.4 as the following example demonstrates.

Example 4.2.2 Let us consider two functions

$$f(x) = g(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Let us take the A-approximation $R = \{<(1/2), 1>, <(1/4), (1/2)>, <(1/8), (1/4)>, \ldots, <(1/2)^{2n}, (1/2)^{2n-1}>, \ldots\}$. Then $\partial/_{\partial R}f(0) = \partial/_{\partial R}g(0) = 1$, $\partial/_{\partial R}(f\cdot g)(0) = \lim_{n\to\infty}((((1/2)^{2n-1} - (1/2)^{2n})^2)/((1/2)^{2n-1} - (1/2)^{2n})) = \lim_{n\to\infty}((1/2)^{2n-1} - (1/2)^{2n}) = 0$, while $g(0)\cdot\partial/_{\partial R}f(0) + f(0)\cdot\partial/_{\partial R}g(0) = 1\cdot 1 + 1\cdot 1 = 1$, i.e., $\partial/_{\partial R}(f\cdot g)(0) \neq g(0)\cdot\partial/_{\partial R}f(0) + f(0)\cdot\partial/_{\partial R}g(0)$.

Theorem 4.2.4 tells us that continuity of functions f and g at a point a is sufficient for validity of the Product Rule for any sequential partial derivative at a. So, it is natural to ask the question whether this condition is also necessary, i.e., we have the following open problem.

Problem 4.2.1 *Does validity of the Product Rule for any sequential partial derivative of the product of the given two functions f and g at a point a implies continuity of functions f and g at this point*?

Theorem 4.2.5 *If functions f and g are continuous at a point (real number) a and g is not equal to* 0 *at a point* $a \in \mathbf{R}$ *and all points from an* A*-approximation* $R = \{ <a_i, b_i>; i \in \omega\}$ *of a, then we have*

$$\partial/_{\partial R}\,(f/g)(a) = [g(a) \cdot \partial/_{\partial R}\, f(a) - f(a) \cdot \partial/_{\partial R}\, g(a)]/(g(a))^2$$

Proof By definition, $\partial/_{\partial R}\,(f/g)(a) = \mathrm{Hn}(c_n)_{n\in\omega}$ where $c_n = ((f(b_n)/g(b_n)) - (f(a_n)/g(a_n)))/(b_n - a_n) = \Delta_n{}^R(f/g)/\Delta_n{}^R x$ for all $n = 1, 2, 3, \ldots$. Note that we may divide by $g(b_n)$ and $g(a_n)$ because g is not equal to 0 at all points from R.

Let us consider one element of the sequence that defines the derivative $\partial/_{\partial R}\,(f/g)(a)$.

$$\begin{aligned}\Delta_n{}^R(f/g)/\Delta_n{}^R x &= \frac{(f(b_n)/g(b_n)) - (f(a_n)/g(a_n))}{b_n - a_n} = \frac{f(b_n)g(a_n) - f(a_n)g(b_n)}{g(a_n)g(b_n)(b_n - a_n)} \\ &= \frac{f(b_n)g(a_n) - f(a_n)g(a_n) + f(a_n)g(a_n) - f(a_n)g(b_n)}{g(a_n)g(b_n)(b_n - a_n)} \\ &= \frac{f(b_n)g(a_n) - f(a_n)g(a_n)}{g(a_n)g(b_n)(b_n - a_n)} - \frac{f(a_n)g(b_n) - f(a_n)g(a_n)}{g(a_n)g(b_n)(b_n - a_n)} \\ &= \frac{(f(b_n) - f(a_n))/(b_n - a_n)}{g(a_n)g(b_n)} g(a_n) - f(b_n)\frac{(g(b_n) - g(a_n))/(b_n - a_n)}{g(a_n)g(b_n)}\end{aligned}$$

As this is true for all $n \in \omega$, we have

$$\mathrm{Hn}(\Delta_n{}^R(f/g)/\Delta_n{}^R x)_{n\in\omega} = \mathrm{Hn}(d_n)_{n\in\omega}$$

where $d_n = ((g(a_n)(\Delta_n{}^R(f)/\Delta_n{}^R x) + (f(b_n)(\Delta_n{}^R(g)/\Delta_n{}^R x)/(g(a_n)g(b_n))$

Continuity of f and g at the point a implies that both functions are bounded in some neighborhood of a (cf. Burgin 2008a). As g is not equal to 0 at the point a and is continuous at this point, sequences $\{g(a_n); n = 1, 2, 3, \ldots\}$ and $\{g(b_n); n = 1, 2, 3, \ldots\}$ are separated from 0. Thus, by Theorems 2.2.2 and 2.2.3, we have

$$\begin{aligned}\mathrm{Hn}(\Delta_n{}^R(f/g)/\Delta_n{}^R x)_{n\in\omega} = \mathrm{Hn}(1/(g(a_n)g(b_n)))_{n\in\omega} \cdot (\mathrm{Hn}(f(a_n)(\Delta_n{}^R(g)/\Delta_n{}^R x)_{n\in\omega} \\ + \mathrm{Hn}(g(a_n)(\Delta_n{}^R(f)/\Delta_n{}^R x)_{n\in\omega})\end{aligned}$$

As f is continuous at the point a and $\lim_{n\to\infty} b_n = a$, we have $\mathrm{Hn}(f(a_n))_{n\in\omega} = \mathrm{Hn}(f(b_n))_{n\in\omega} = f(a)$.

As g is continuous at the point a and $\lim_{n\to\infty} a_n = a$, we have $\mathrm{Hn}(g(b_n))_{n\in\omega} = \mathrm{Hn}(g(a_n))_{n\in\omega} = g(a)$ and $\mathrm{Hn}(1/(g(a_n)g(b_n)))_{n\in\omega} = 1/(g(a)g(b))$.

Consequently,

$$\begin{aligned}\Delta_n{}^R(f/g)/\Delta_n{}^R x &= (1/(g(a)g(a))) \cdot \left(g(a)\frac{f(b_n) - f(a_n)}{b_n - a_n} - f(a)\frac{g(b_n) - g(a_n)}{b_n - a_n}\right) \\ &= (1/(g(a)^2) \cdot (g(a)(\Delta_n{}^R(f)/\Delta_n{}^R x) - f(a)(\Delta_n{}^R(g)/\Delta_n{}^R x))\end{aligned}$$

Thus, by Theorem 2.2.3,

$$\begin{aligned}\partial/_{\partial R}(f/g)(a) &= \mathrm{Hn}({\Delta_n}^R(f/g)/{\Delta_n}^R x)_{n\in\omega}\\ &= (1/(g(a)^2)\cdot(\mathrm{Hn}(f(a_n)({\Delta_n}^R(g)/{\Delta_n}^R x)_{n\in\omega}+\mathrm{Hn}(g\\ &\quad\times(a_n)({\Delta_n}^R(f)/{\Delta_n}^R x)_{n\in\omega})\\ &= (g(a)\cdot\partial/_{\partial R}f(a)-f(a)\cdot\partial/_{\partial R}g(a))/g(a)^2\end{aligned}$$

Theorem is proved.

We see that sequential partial derivatives have many properties of conventional derivatives. At the same time, there are many properties of sequential partial derivatives that are essentially different from properties of conventional derivatives. Here is one of such results.

Theorem 4.2.6 *There is a function f(x) and a real number a such that for any real hypernumber* α*, there is an* A*-approximation R such that* $\partial/_{\partial R}f(a)=\alpha$.

Proof Let us assume that $\alpha=\mathrm{Hn}(a_i)_{x\in\omega}$. To build such a function $f(x)$ and A-approximation $R=\{<c_i,d_i>;i=1,2,3,\ldots\}$ of a point a, let us consider the following sets $\boldsymbol{Q}$ of all rational numbers and $\boldsymbol{Q}_p=\sqrt{p}\cdot\boldsymbol{Q}$ for all prime numbers $p=2,3,5,\ldots$. Properties of real numbers imply that these sets do not intersect and that these sets, the complement of their union $\boldsymbol{R}_0=\boldsymbol{R}/(\boldsymbol{Q}\bigcup(\bigcup_p\boldsymbol{Q}_p))$ and $\boldsymbol{Q}_p$ are dense in $\boldsymbol{R}$. This allows us to define the following function and take 0 as the point a.

$$f(x)=\begin{cases}(1/p)\cdot|x| & \text{when } x\in\boldsymbol{Q}_p\\ |x| & \text{when } x\in\boldsymbol{Q}\\ 0 & \text{for all other real numbers } x,\ \text{i.e., when } x\in\boldsymbol{R}_0\end{cases}$$

Now we can construct the necessary A-approximation $R=\{<c_i,d_i>;$ $i=1,2,3,\ldots\}$ of the point 0. At first, we assume that the number a_1 is positive.

As the second point d_1 in the first pair $<c_1,d_1>$ in the A-approximation R, we take a number d_1 from $\boldsymbol{Q}_{p_1}$ such that $0\le 1-d_1<1/10$. It is possible to find such a number because $\boldsymbol{Q}_{p_1}$ is dense in $\boldsymbol{R}$. Note that $d_1<1$ because $1\in\boldsymbol{Q}$.

It is possible to find a prime number p_1 such that $1/p_1<a_1$. Then taking a number $u_1=(d_1a_1-(1/p_1)(d_1))/a_1$, we see that for $x_1=u_1$ and $x_2=d_1$, we have $\Delta_{f,1}/\Delta_{x,1}=\Delta_f/\Delta_x=((1/p_1)(d_1)-0)/(d_1-u_1)=a_1$ when u_1 belongs to $\boldsymbol{R}_0$. In this case, we regard u_1 as the first point c_1 in the first pair $<c_1,d_1>$ in the A-approximation R.

However, it is possible that u_1 does not belong to $\boldsymbol{R}_0$. In this case, as $\boldsymbol{R}_0$ is dense in $\boldsymbol{R}$, there is a number c_1 from $\boldsymbol{R}_0$ such that $0\le c_1-u_1<\varepsilon_1$ where $\varepsilon_1<(1/2)(d_1/a_1)$. Then we take $<c_1,d_1>$ as the first pair in the A-approximation R. Note that $c_1<u_1+\varepsilon_1$.

Now let us estimate the distance between the ratio $\Delta_{f,1}/\Delta_{x,1}=\Delta_f/\Delta_x$ for $x_1=c_1$ and $x_2=d_1$ and the number a_1.

$$
\begin{aligned}
|(\Delta_{f,1}/\Delta_{x,1}) - a_1| &= |(\Delta_{f,1}/\Delta_{x,1}) - ((1/p_1)(d_1) - 0)/(d_1 - u_1)| \\
&= |(((1/p_1)(d_1) - 0)/(d_1 - c_1)) - (((1/p_1)(d_1) - 0)/(d_1 - u_1))| \\
&= |((1/p_1)(d_1)/(d_1 - c_1)) - ((1/p_1)(d_1)/(d_1 - u_1))| \\
&= |((1/p_1)(d_1)(d_1 - u_1) - (1/p_1)(d_1)(d_1 - c_1))/((d_1 - u_1)(d_1 - c_1))| \\
&= |((1/p_1)(d_1)u_1 - (1/p_1)(d_1)c_1)/((d_1 - u_1)(d_1 - c_1))| \\
&= ((1/p_1)(d_1)(c_1 - u_1))/((d_1 - u_1)(d_1 - c_1)) \\
&< ((1/p_1)(d_1)\varepsilon_1)/((d_1 - u_1)(d_1 - c_1)) \\
&= ((1/p_1)(d_1)\varepsilon_1)/((d_1 - u_1)d_1 - (d_1 - u_1)c_1) \\
&< ((1/p_1)(d_1)\varepsilon_1)/((d_1 - u_1)d_1)
\end{aligned}
$$

By construction, $u_1 = (d_1 a_1 - (1/p_1)(d_1))/a_1$. Thus, we have

$$
(d_1 - u_1)d_1 = d_1 d_1 - u_1 d_1 = {d_1}^2 - {d_1}^2 + ((1/p_1)({d_1}^2))/a_1 = ((1/p_1)({d_1}^2))/a_1
$$

Consequently,

$$
\begin{aligned}
&|(\Delta_{f,1}/\Delta_{x,1}) - a_1| < ((1/p_1)(d_1)\varepsilon_1)/((d_1 - u_1)d_1) \\
&\quad = ((1/p_1)(d_1)\varepsilon_1)/(((1/p_1)({d_1}^2))/a_1) = ((1/p_1)(d_1 a_1)\varepsilon_1)/((1/p_1)({d_1}^2)) \\
&\quad = (a_1\varepsilon_1)/d_1 < \frac{1}{2}
\end{aligned}
$$

As a result,

$$
|(\Delta_{f,1}/\Delta_{x,1}) - a_1| < \frac{1}{2}
$$

If the number a_1 is negative, we find the pair $< c'_1, d'_1 >$ for the number $|a_1|$ by the procedure described above. Then we take $< -d'_1, -c'_1 >$ as the first pair $< c_1, d_1 >$ in the A-approximation R. In this case, we also have

$$
\begin{aligned}
|(\Delta_{f,1}/\Delta_{x,1}) - a_1| &= |(\Delta_{f,1}/\Delta_{x,1}) - ((1/p_1)(d_1) - 0)/(d_1 - u_1)| \\
&= |(1/p_1)(d_1)/(d_1 - c_1) - (1/p_1)(d_1)/(d_1 \\
&\quad - u_1)| < (a_1\varepsilon_1)/|d_1| < \frac{1}{2}
\end{aligned}
$$

and

$$
|(\Delta_{f,1}/\Delta_{x,1}) - a_1| < \frac{1}{2}
$$

If $a_1 = 0$, then we take a number d_1 from $\boldsymbol{R}_0$ such that $0 \leq 1 - d_1 < 1/10$. We also take a number c_1 from $\boldsymbol{R}_0$ such that $0 < d_1 - c_1 < 1/10$. It is possible to find such numbers because $\boldsymbol{R}_0$ is dense in $\boldsymbol{R}$. Note that $d_1 < 1$ and $c_1 < d_1$ because

$1 \in \boldsymbol{Q}$. Then we take $<c_1, d_1>$ as the first pair in the A-approximation R. In this case, $\Delta_{f,1}/\Delta_{x,1} = a_1$. Thus, in all cases, we have

$$|(\Delta_{f,1}/\Delta_{x,1}) - a_1| < \frac{1}{2}$$

A similar procedure is used for finding the second pair $<c_2, d_2>$ in R. At first, as before, we assume that the number a_2 is positive. Then taking the number a_2, we define $p_2 = p_1$ when $1/p_1 < a_2$ or find a prime number p_2 such that $1/p_2 < a_2$. As the second point d_2 in the second pair $<c_2, d_2>$ in R, we take a number d_2 from $\boldsymbol{Q}_{p_2}$ such that $0 \leq |c_1| - d_2 < (1/10) \cdot |c_1|$. It is possible to find such a number because $\boldsymbol{Q}_{p_2}$ is dense in $\boldsymbol{R}$. Note that $0 < d_2 \leq |c_1|$.

Then taking a number $u_2 = (d_2a_2 - (1/p_2)(d_2))/a_2$, we see that for $x_1 = u_2$ and $x_2 = d_2$, we have $\Delta_{f,2}/\Delta_{x,2} = \Delta_f/\Delta_x = ((1/p_2)(d_2) - 0)/(d_2 - u_2) = a_2$ when u_2 belongs to $\boldsymbol{R}_0$. In this case, we regard u_2 as the first point c_2 in the second pair $<c_2, d_2>$ in the A-approximation R. Note that $0 < u_2 \leq d_2$.

However, it is possible that u_2 does not belong to $\boldsymbol{R}_0$. In this case, as $\boldsymbol{R}_0$ is dense in $\boldsymbol{R}$, there is a number c_2 from $\boldsymbol{R}_0$ such that $0 \leq c_2 - u_2 < \varepsilon_2$ where $\varepsilon_2 < (1/4)(d_2/a_2)$. Then we take $<c_2, d_2>$ as the second pair in the A-approximation R. Note that $c_2 < u_2 + \varepsilon_2$.

Estimating as before the distance between the ratio $\Delta_{f,2}/\Delta_{x,2} = \Delta_f/\Delta_x$ for $x_1 = c_2$ and $x_2 = d_2$ and the number a_2, we see that

$$|(\Delta_{f,2}/\Delta_{x,2}) - a_2| < \frac{1}{4}$$

If the number a_2 is negative, we find the pair $<c'_2, d'_2>$ for the number $|a_2|$ by the procedure described above. Then we take $< -d'_2, -c'_2>$ as the second pair $<c_2, d_2>$ in the A-approximation R. Note that both numbers c_2 and d_2 are negative. In this case, we also have

$$|(\Delta_{f,2}/\Delta_{x,2}) - a_2| < \frac{1}{4}$$

If $a_2 = 0$, then we take a number d_2 from $\boldsymbol{R}_0$ such that $0 \leq |c_1| - d_2 < (1/10)|c_1|$. We also take a number c_2 from $\boldsymbol{R}_0$ such that $0 < d_2 - c_2 < (1/10)|c_1|$. It is possible to find such numbers because $\boldsymbol{R}_0$ is dense in $\boldsymbol{R}$. Note that $d_2 < c_1$ and $c_2 < d_2$. Then we take $<c_2, d_2>$ as the second pair in the A-approximation R. In this case, $\Delta_{f,2}/\Delta_{x,2} = a_2$. Thus, in all cases, we have

$$|(\Delta_{f,2}/\Delta_{x,2}) - a_2| < \frac{1}{4}$$

The general procedure of choosing pairs in R has the following form. After making $i - 1$ similar steps, we take the number a_i. Assuming that a_i is positive, we define $p_i = p_{i-1}$ when $1/p_{i-1} < a_2$ or find a prime number p_i such that $1/p_i < a_i$.

As the second point d_i in the ith pair $<c_i, d_i>$ in R, we take a number d_i from $\boldsymbol{Q}_{p_i}$ such that $0 \leq |c_{i-1}| - d_i < (1/10) \cdot |c_{i-1}|$. It is possible to find such a number because $\boldsymbol{Q}_{p_i}$ is dense in $\boldsymbol{R}$. Note that $0 < d_i \leq |c_{i-1}|$.

Then taking a number $u_i = (d_i a_i - (1/p_i)(d_i))/a_i$, we see that for $x_1 = u_i$ and $x_2 = d_i$, we have $\Delta_{f,i}/\Delta_{x,i} = \Delta_f/\Delta_x = ((1/p_i)(d_i) - 0)/(d_i - u_i) = a_i$ when u_i belongs to $\boldsymbol{R}_0$. In this case, we regard u_i as the first point c_i in the ith pair $<c_i, d_i>$ in the A-approximation R. Note that $0 < u_i \leq d_i$.

However, it is possible that u_i does not belong to $\boldsymbol{R}_0$. In this case, as $\boldsymbol{R}_0$ is dense in $\boldsymbol{R}$, there is a number c_i from $\boldsymbol{R}_0$ such that $0 \leq c_i - u_i < \varepsilon_i$ where $\varepsilon_i < ((1/2)^i)(d_i/a_i)$. Then we take $<c_i, d_i>$ as the ith pair in the A-approximation R. Note that $c_i < u_i + \varepsilon_i$.

Now let us estimate the distance between the ratio $\Delta_{f,i}/\Delta_{x,i} = \Delta_f/\Delta_x$ for $x_1 = c_i$ and $x_2 = d_i$ and the number a_i, taking into account that $c_i - u_i < \varepsilon_i$.

$$\begin{aligned}
|(\Delta_{f,i}/\Delta_{x,i}) - a_i| &= |(\Delta_{f,i}/\Delta_{x,i})_x - (((1/p_i)(d_i) - 0)/(d_i - u_i))| \\
&= |(((1/p_i)(d_i) - 0)/(d_i - c_i)) - (((1/p_i)(d_i) - 0)/(d_i - u_i))| \\
&= |((1/p_i)(d_i)/(d_i - c_i)) - ((1/p_i)(d_i)/(d_i - u_i))| \\
&= |((1/p_i)(d_i)(d_i - u_i) - (1/p_i)(d_i)(d_i - c_i))/((d_i - u_i)(d_i - c_i))| \\
&= |((1/p_i)(d_i)u_i - (1/p_i)(d_i)c_i)/((d_i - u_i)(d_i - c_i))| \\
&= ((1/p_i)(d_i)(c_i - u_i))/((d_i - u_i)(d_i - c_i)) \\
&< ((1/p_i)(d_i\varepsilon_i))/((d_i - u_i)(d_i - c_i)) \\
&= ((1/p_i)(d_i\varepsilon_i))/((d_i - u_i)d_i - (d_i - u_i)c_i) \\
&< ((1/p_i)(d_i)\varepsilon_i)/((d_i - u_i)d_i)
\end{aligned}$$

By construction, $u_i = (d_i a_i - (1/p_i)(d_i))/a_i = d_i - ((1/p_i)(d_i))/a_i$. Thus, we have

$$(d_i - u_i)d_i = d_i d_i - u_i d_i = d_i^2 - d_i^2 + ((1/p_i)(d_i^2)/a_i = ((1/p_i)(d_i^2)/a_i$$

Consequently,

$$|(\Delta_{f,i}/\Delta_{x,i}) - a_i| < ((1/p_i)(d_i)\varepsilon_i)/((d_i - u_i)d_i) = ((1/p_i)(d_i)\varepsilon_i)/((1/p_i)(d_i^2)/a_i)$$

$$= ((1/p_i)(d_i a_i)\varepsilon_i)/((1/p_i)(d_i^2)) = (a_i\varepsilon_i)/d_i < \frac{1}{2}^i$$

As a result,

$$|(\Delta_{f,i}/\Delta_{x,i}) - a_i| < \frac{1}{2}^i$$

If the number a_i is negative, we find the pair $<c'_i, d'_i>$ for the number $|a_i|$ by the procedure described above. Then we take $<-d'_i, -c'_i>$ as the first pair $<c_i, d_i>$

in the A-approximation R. Note that both numbers c_i and d_i are negative. In this case, we also have

$$|(\Delta_{f,i}/\Delta_{x,i}) - a_i| = |(\Delta_{f,i}/\Delta_{x,i}) - ((1/p_i)(d_i) - 0)/(d_i - u_i)|$$
$$= |(1/p_i)(d_i)/(d_i - c_i) - (1/p_i)(d_i)/(d_i - u_i)| < (a_i \varepsilon_i)/|d_i| < \frac{1}{2}^i$$

and

$$|(\Delta_{f,i}/\Delta_{x,i}) - a_i| < \frac{1}{2}^i$$

If $a_i = 0$, then we take a number d_i from $\boldsymbol{R}_0$ such that $0 \leq |c_{i-1}| - d_i < (1/10)|c_{i-1}|$. We also take a number c_i from $\boldsymbol{R}_0$ such that $0 < d_i - c_i < (1/10)|c_{i-1}|$. It is possible to find such numbers because $\boldsymbol{R}_0$ is dense in $\boldsymbol{R}$. Note that $d_i < |c_{i-1}|$ and $0 < c_i < d_i$. Then we take $< c_i, d_i >$ as the first pair in the A-approximation R. In this case, $\Delta_{f,i}/\Delta_{x,i} = a_i$. Thus, in all cases, we have

$$|(\Delta_{f,i}/\Delta_{x,i}) - a_i| < \frac{1}{2}^i$$

We cannot perform infinitely many steps, but applying the Constructive Principle of Induction (cf. Appendix), we prove existence of a set $R = \{< c_i, d_i >; i = 1, 2, 3, \ldots\}$ with necessary properties.

Now let us take the partial derivative $\partial/_{\partial R} f(a) = \mathrm{Hn}(\Delta_{f,i}/\Delta_{x,i})_{i\in\omega}$ and compare it with the hypernumber $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$. By construction, we have

$$\lim_{i\to\infty}|(\Delta_{f,i}/\Delta_{x,i}) - a_i| = 0$$

It means that $\partial/_{\partial R} f(a) = \alpha$.

Theorem is proved because α is an arbitrary real hypernumber.

This property of sequential partial derivatives is essentially different from properties of conventional derivatives because if such a derivative of a function at a point exists, it is unique.

The function $f(x)$ that is used in the proof of Theorem 4.2.6 is essentially discontinuous. Thus, it is natural to ask the following question.

Problem 4.2.2 *Is there a continuous function $f(x)$ such that for some real number a and for any real hypernumber α, there is an* A-*approximation R of a such that $\partial/_{\partial R} f(a) = \alpha$?*

Definition 4.2.2

(a) A partial derivative $\partial/_{\partial R} f(a)$ is called *one-sided* (*two-sided*) if $R \in rappr(a) \bigcup lappr(a)$ ($R \in bappr(a)$).

(b) $\partial/_{\partial R} f(a)$ is called *right* (*left*) *partial derivative* of f at a if $R \in rappr(a)$ $(R \in lappr(a))$.
(c) A one-sided partial derivative $\partial/_{\partial R} f(a)$ is called *fundamental* if R is a stable A-approximation of a.
(d) A one-sided partial derivative $\partial/_{\partial R} f(a)$ is called *almost fundamental* if R is an almost stable A-approximation of a.

Lemma 4.1.5 implies that any fundamental one-sided partial derivative is almost fundamental. Moreover, in some cases, the inverse statement is also true.

Let us assume that all one-sided partial derivatives of f at a are bounded.

Lemma 4.2.2 *A one-sided partial derivative $\partial/_{\partial R} f(a)$ with respect to an* A-*approximation R is almost fundamental if and only if it is equal to a fundamental partial derivative $\partial/_{\partial Q} f(a)$ with respect to some* A-*approximation Q.*

Proof Sufficiency follows directly from Lemma 4.1.5. So, it is necessary to prove only necessity.

Let us take an almost fundamental right partial derivative $\partial/_{\partial R} f(a)$ for an almost stable right A-approximation $R = \{<a_i, b_i>; i \in \omega\}$ of the point a. There are two possible cases: $\partial/_{\partial R} f(a) = 0$ and $\partial/_{\partial R} f(a) \neq 0$.

1. At first, let us assume that $\partial/_{\partial R} f(a) = \mathrm{Hn}((f(b_i) - f(a_i))/(b_i - a_i))_{i\in\omega} = 0$ and consider the stable A-approximation

   $Q = \{<a_i, b_i> \text{ where } <a_i, b_i> \in R \text{ for all } i = 1, 2, 3, \ldots\}$. Then for all $i = 1, 2, 3, \ldots,$

   we have

$$\begin{aligned}(f(b_i) - f(a))/(b_i - a) &= (f(b_i) - f(a_i) + f(a_i) - f(a))/((b_i - a_i) + (a_i - a)) \\ &= (f(b_i) - f(a_i))/((b_i - a_i) + (a_i - a)) + (f(a_i) \\ &\quad - f(a))/((b_i - a_i) + (a_i - a))\end{aligned}$$

   and by Theorem 4.2.2, we have

$$\begin{aligned}\partial/_{\partial R} f(a) &= \mathrm{Hn}((f(b_i) - f(a))/(b_i - a))_{i\in\omega} \\ &= \mathrm{Hn}((f(b_i) - f(a_i))/((b_i - a_i))_{i\in\omega} + (a_i - a)) \\ &\quad + \mathrm{Hn}((f(a_i) - f(a))/((b_i - a_i) + (a_i - a)))_{i\in\omega}\end{aligned}$$

   In this sum, $\mathrm{Hn}((f(b_i) - f(a_i))/((b_i - a_i) + (a_i - a)))_{i\in\omega} \leq 0$ because $\lim_{i\to\infty} (f(b_i) - f(a_i))/(b_i - a_i) = 0$.

   The condition of the lemma implies existence of a number $C \in \boldsymbol{R}^+$ such that $|(f(a_i) - f(a))/(a_i - a)| \leq C$ when $i \geq i_1$. Then $|f(a_i) - f(a)| \leq C|a_i - a|$ and for the second summand $|(f(a_i) - f(a))/((b_i - a_i) + (a_i - a))| \leq C|a_i - a|/|(b_i - a_i) - (a_i - a)| \leq C(|a_i - a|/(b_i - a)) \to 0$. Thus, $\partial/_{\partial Q} f(a) = \mathrm{Hn}((f(b_i) - f(a))/(b_i - a))_{i\in\omega} = 0$.

2. Let $\partial/_{\partial R} f(a) = \alpha \neq 0$. Then

$$\begin{aligned}
\partial/_{\partial R} f(a) &= \text{Hn}((f(b_i) - f(a_i))/(b_i - a_i))_{i\in\omega} \\
&= \text{Hn}(f(b_i) - f(a)) - (f(a_i) - f(a)))/(b_i - a_i))_{i\in\omega} \\
&= \text{Hn}((f(b_i) - f(a))/(b_i - a_i)) - (f(a_i) - f(a))/(b_i - a_i)))_{i\in\omega} \\
&= \text{Hn}((f(a_i) - f(a))/(b_i - a_i))_{i\in\omega}
\end{aligned}$$

because

$$\begin{aligned}
&\text{Hn}((f(a_i) - f(a))/(b_i - a_i))_{i\in\omega} = \text{Hn}((f(a_i) - f(a))/(a_i - a)) \cdot ((a_i - a)/(b_i - a_i)))_{i\in\omega} \\
&\quad = \text{Hn}((f(a_i) - f(a))/(a_i - a))_{i\in\omega} \cdot \text{Hn}((a_i - a)/(b_i - a_i))_{i\in\omega} \\
&\quad = \text{Hn}(((f(a_i) - f(a))/(b_i - a_i)) \cdot 0)_{i\in\omega} = 0
\end{aligned}$$

as $\text{Hn}((f(a_i) - f(a))/(b_i - a_i))_{i\in\omega} = \partial/_{\partial K} f(a) \leq C \in \boldsymbol{R}$ where $Q = \{<a, a_i>; i = 1, 2, 3, \ldots\}$ and (cf. Definition 4.2.1) $\text{Hn}((a_i - a)/(b_i - a_i))_{i\in\omega} = 0$.

At the same time,

$$\begin{aligned}
\partial/_{\partial Q} f(a) &= \text{Hn}((f(b_i) - f(a))/(b_i - a))_{i\in\omega} \\
&= \text{Hn}((f(b_i) - f(a))/(b_i - a))_{i\in\omega} \cdot \text{Hn}((b_i - a)/(b_i - a_i))_{i\in\omega} \\
&= \text{Hn}((f(b_i) - f(a))/(b_i - a_i))_{i\in\omega}
\end{aligned}$$

because $\lim_{i\to\infty}((b_i - a)/(b_i - a_i)) = \lim_{i\to\infty}((b_i - a_i + a_i - a)/(b_i - a_i)) = \lim_{i\to\infty}(1 + (a_i - a)/(b_i - a_i)) = 1 + \lim_{i\to\infty}((a_i - a)/(b_i - a_i)) = 1 + 0 = 1$ and consequently, $\text{Hn}((b_i - a)/(b_i - a_i))_{i\in\omega} = 1$. So, $\partial/_{\partial R} f(a) = \partial/_{\partial Q} f(a)$.

The proof for a left almost fundamental partial derivative $\partial/_{\partial T} f(a)$, is similar.

Lemma is proved.

Remark 4.2.2 The condition of Lemma 4.2.2 that all one-sided partial derivatives are bounded is essential. This is demonstrated by the function $g(x)$, which is equal to 1 when $x = 1$ and equal to 0 at all other points of $\boldsymbol{R}$. For any stable approximation R of the point 1, the partial derivative $\partial/_{\partial R} g(a)$ is an infinite hypernumber. At the same time, for any non-stable one sided approximation R in which $a_i \neq a$ and $b_i \neq a$ for all $i \in \omega$, the partial derivative $\partial/_{\partial R} g(x)$ is equal to 0.

Let $f: \boldsymbol{R} \to \boldsymbol{R}$ be a continuous function, R, $Q \in bappr(a)$, $\alpha, \beta \in \boldsymbol{R}_\omega$ and $\alpha < \beta$. Then the following theorem about the intermediate value of partial derivatives is valid.

Theorem 4.2.7 (*Local Intermediate Value Theorem*) *If* $\alpha = \partial/_{\partial R} f(a)$, $\beta = \partial/_{\partial Q} f(a)$ *and* η *is a real hypernumber for which the inequalities* $\alpha < \eta < \beta$ *are valid, then a two-sided approximation* $P \in bappr(a)$ *exists such that* $\partial/_{\partial P} f(a) = \eta$.

Proof Let us assume that $R = \{<v_i, u_i>; i \in \omega\}$, $Q = \{<z_i, w_i>; i \in \omega\}$ are two-sided A-approximations of the point $a \in \boldsymbol{R}$, $\partial/_{\partial R} f(a) = \alpha$, $\partial/_{\partial Q} f(a) = \beta$, and $\alpha < \beta$. By the definition $\partial/_{\partial R} f(a) = \text{Hn}(a_i)_{i\in\omega}$ where $a_i = (f(u_i) - f(v_i))/(u_i - v_i)$ and $\partial/_{\partial Q} f(a) = \text{Hn}(b_i)_{i\in\omega}$ where $b_i = (f(w_i) - f(z_i))/(w_i - z_i)$. As $\alpha < \beta$ then by the definition of order relation on $\boldsymbol{R}_\omega$, there is a natural number m such that $a_i < b_i$ for all $i > m$.

Let us demonstrate validity of the following statement (A):

If $(f(u) - f(v))/(u - v) < (f(w) - f(z))/(w - z)$, $v < a < u$ and $z < a < w$, then for any number d that satisfies the inequality $a = (f(u) - f(v))/(u - v) < d < (f(w) - f(z))/(w - z) = b$, there are such points t and r for which $(f(t) - f(r))/(t - r) = d$ and $\min\{v, z\} \leq r < a < t \leq \max\{u, w\}$.

If $e = (f(u) - f(z))/(u - z)$, then there are four possibilities:
(1) $e < a < d < b$; (2) $a < e \leq d < b$; (3) $a < d \leq e < b$; (4) $a < d < b < e$.

If $e = d$, then the necessary statement is valid by the definition of e and d. If $e < d$, then at first we analyze the cases 1 and 2. For all $y \in [u, w]$ the inequality $y < z$ is valid and, as a consequence, $y - z > 0$ for all such y. Consequently, the function $g(y) = (f(y) - f(z))/(y - z)$ is continuous on the interval $[u, w]$ (or $[w, u]$ when $w < u$). By the definition of $g(y)$, we have $g(u) = e < d < g(w) = b$. By the Intermediate Value Theorem for continuous functions (Ross 1996; Burgin 2008a), there is a number $q \in [u, w]$ such that $g(q) = d$. Then by the definition of $g(y)$, we have $(f(q) - f(z))/(q - z) = d$. So, taking r equal to q and t equal to z, we obtain the necessary statement. Cases 3 and 4 are treated in a similar way. The statement (A) is proved.

To continue the proof of the theorem, we remind that the inequalities $\alpha < \eta < \beta$ are valid for a hypernumber $\eta = \mathrm{Hn}(d_i)_{i \in \omega}$. Then, by the definition of the order relation on $\boldsymbol{R}_\omega$, all inequalities $a_i < d_i < b_i$ are valid for all $i > k$ where k is some natural number.

Using the proved result, it is possible to find points $r_i < a < t_i$ such that $a_i < d_i = (f(t_i) - f(r_i))/(t_i - r_i) < b_i$ for all $i > k$. Let $P = \{<r_i, t_i>; i \in \omega\}$. By the definition of the points r_i, t_i the set of pairs P is a two-sided approximation of the point a, and $\partial/_{\partial P} f(a) = \eta$.

Theorem is proved.

Corollary 4.2.5 *If b, d, and p are real numbers for which the inequalities $d < p < b$ are valid and $d = \partial/_{\partial Q} f(a)$, $b = \partial/_{\partial R} f(a)$, then an* A-*approximation $P \in bappr(a)$ exists such that $\partial/_{\partial P} f(a) = p$.*

Theorem 4.2.8 (*Extended Rolle Theorem*) *If a function $f(x)$ is continuous on $[a, b]$ and $f(a) = f(b)$, then there is (at least one) point $c \in [a, b]$ and an* A-*approximation R of c such that $\partial/_{\partial R} f(c) = 0$.*

Proof Let us take a continuous function $f(x)$ in the interval $[a, b]$. By properties of continuous functions, $f(x)$ assumes its maximum and minimum values at some points c and d in $[a, b]$ (Ross 1996; Burgin 2008a). Then there are two options: either both c and d are endpoints of the interval $[a, b]$ or at least, one of them (say, c) is an inner point of the interval $[a, b]$.

In the first case, maximum and minimum values of $f(x)$ coincide because $f(a) = f(b)$. Consequently, $f(x)$ is a constant function. Thus, the statement of the theorem is true because $f(x)$ has the derivative at any inner point of the interval $[a, b]$ and this derivative is always equal to zero.

Now we have to treat the case when a maximum or minimum is reached by $f(x)$ at an inner point c of the interval $[a, b]$. We assume that $f(c)$ is the maximum of $f(x)$. In this situation, there are also two options: either there are infinitely many points in $[a, b]$ where $f(x)$ reaches its maximum or $f(c) > f(x)$ for all x in some neighborhood $(c - h, c + h)$ of the point c.

In the first case, there is a converging sequence $\{c_i; i = 1, 2, 3, \ldots\}$ such that all values $f(c_i)$ are equal to the maximum of $f(x)$ in $[a, b]$. If $d = \lim_{i\to\infty} c_i$, then $f(d)$ is also the maximum of $f(x)$ in $[a, b]$. This allows us to build the A-approximation $R = \{<c_i, d>; i = 1, 2, 3, \ldots\}$ of the point d. As $f(c_i) - f(d) = 0$ for all $i = 1, 2, 3, \ldots$, we have $\partial/_{\partial R} f(d) = 0$. So, the statement of the theorem is once more true.

Now let us consider the case when $f(c) > f(x)$ for all x in some neighborhood $(c - h, c + h)$. To find the first pair $<c_1, d_1>$ in the A-approximation $R = \{<c_i, d>; i = 1, 2, 3, \ldots\}$ of the point c, we compare values $f(c - h)$ and $f(c + h)$. If these values are equal, then we take $c_1 = c - h$ and $d_1 = c + h$. If one of these values is larger than another one, e.g., $f(c - h) > f(c + h)$, then by the Intermediate Value Theorem for continuous functions (Ross 1996; Burgin 2008a), there is a point c_1 between $c + h$ and c such that $f(c) > f(c_1) = f(c - h) > f(c + h)$. Then we take $d_1 = c + h$ and define $<c_1, d_1>$ as the first pair in the A-approximation R. By construction, $(f(d_1) - f(c_1))/(d_1 - c_1) = f(d_1) - f(c_1) = 0$.

As $f(x)$ is a continuous function, there is a neighborhood $(c - h_1, c + h_1)$ of the point c such that $h_1 = (1/2)h$ and $f(c) - f(x) < (1/2)f(c_1)$ for all x in $(c - h_1, c + h_1)$. To find the second pair $<c_2, d_2>$ in the A-approximation R, we compare values $f(c - h_1)$ and $f(c + h_1)$. If these values are equal, then we take $c_2 = c - h_1$ and $d_2 = c + h_1$. If one of these values is larger than another one, e.g., $f(c - h_1) > f(c + h_1)$, then by the Intermediate Value Theorem for continuous functions, there is a point c_2 between $c + h_1$ and c such that $f(c) > f(c_2) = f(c - h_1) > f(c + h_1)$. Then we take $d_2 = c + h_1$ and take $<c_2, d_2>$ as the second pair in the A-approximation R. By construction, $(f(d_2) - f(c_2))/(d_2 - c_2) = f(d_2) - f(c_2) = 0$.

Let us assume that $n - 1$ pairs $<c_1, d_1>, \ldots, <c_{n-1}, d_{n-1}>$ from the A-approximation R are constructed in such a way that $(f(d_i) - f(c_i))/(d_i - c_i) = f(d_i) - f(c_i) = 0$ and $h_i = (1/2)h_{i-1}$ for all $i = 1, 2, 3, \ldots, n - 1$. As $f(x)$ is a continuous function, there is a neighborhood $(c - h_n, c + h_n)$ of the point c such that $h_n = (1/2)h_{n-1}$ and $f(c) - f(x) < (1/2)f(c_{n-1})$ for all x in $(c - h_n, c + h_n)$. To find the next pair $<c_n, d_n>$ in the A-approximation R, we compare values $f(c - h_n)$ and $f(c + h_n)$. If these values are equal, then we take $c_n = c - h_n$ and $d_n = c + h_n$. If one of these values is larger than another one, e.g., $f(c - h_n) > f(c + h_n)$, then by the Intermediate Value Theorem for continuous functions, there is a point c_n between $c + h_n$ and c such that $f(c) > f(c_n) = f(c - h_n) > f(c + h_n)$. Then we take $d_n = c + h_n$ and take $<c_n, d_n>$ as the second pair in the A-approximation R. By construction, $(f(d_n) - f(c_n))/(d_n - c_n) = f(d_n) - f(c_n) = 0$.

We cannot perform infinitely many steps, but applying the Constructive Principle of Induction (cf. Appendix), we prove existence of a set $R = \{<c_i, d_i>; i = 1, 2, 3, \ldots\}$ with necessary properties. Indeed, as $c - h_n < c_n < d_n < c + h_n$,

$\lim_{i\to\infty} c_i = \lim_{i\to\infty} d_i = c$, i.e., $R = \{<c_i, d_i>; i = 1, 2, 3, \ldots\}$ is an A-approximation of the point c. In addition, we have

$$\partial/_{\partial R} f(a) = \mathrm{Hn}((f(d_n) - f(c_n))/(d_n - c_n))_{n\in\omega} = \mathrm{Hn}(f(d_n) - f(c_n))_{n\in\omega} = 0$$

Theorem is proved.

This gives us the classical result (Ross 1996; Burgin 2008a).

Corollary 4.2.6 (*Rolle Theorem*) *If a function f(x) is continuous on [a, b], differentiable in (a, b), and f(a) = f(b), then there is (at least one) point* $c \in (a, b)$ *such that* $\partial/_{\partial R} f(c) = 0$.

Remark 4.2.3 The Extended Rolle Theorem essentially extends the scope of the Rolle Theorem because the Rolle Theorem is not valid for many continuous functions as the following example demonstrates.

Example 4.2.3 Let us consider the function $f(x) = \sqrt{|x|}$ on the interval [−1, 1]. We see that $f(x)$ is a continuous function on the interval [−1, 1] and $f(1) = f(-1) = 1$. It is also differentiable at all points but 0. However, there are no points in [−1, 1] where the derivative $f'(x)$ is equal to 0. At the same time, there is an A-approximation R of 0 such that $\partial/_{\partial R} f(0) = 0$.

Remark 4.2.4 The condition that $f(x)$ is a continuous function is essential for validity of the Extended Rolle Theorem as the following example demonstrates.

Example 4.2.4 Let us consider the following function

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{if } 1 \geq x \geq 0 \\ |x| & \text{if } -1 \leq x < 0 \end{cases}$$

We see that $f(x)$ is differentiable at all points but 0, continuous at all points but 0 and $f(1) = f(-1) = 1$. However, there are no points c in [−1, 1] and A-approximations R for which $\partial/_{\partial R} f(c) = 0$.

As in the classical case, the Extended Rolle Theorem allows us to prove the Extended Mean Value Theorem.

Theorem 4.2.9 (*Extended Mean Value Theorem*) *If a function f(x) is continuous on* [*a*, *b*], *then there is (at least one) point* $c \in [a, b]$ *and an* A-*approximation R of c such that* $\partial/_{\partial R} f(c) = (f(b) - f(a))/(b - a)$.

Proof Let us consider the function $g(x) = f(x) - l(x)$ where $l(x) = [(f(b) - f(a))/(b - a)]x + [(bf(a) - af(b))/(b - a)]$. As $l(x)$ is continuous on [a, b], $g(x)$ is also continuous on [a, b]. Besides, we have

$$\begin{aligned} l(a) &= [(f(b) - f(a))/(b - a)]a + [(bf(a) - af(b))/(b - a)] \\ &= [(af(b) - af(a) + bf(a) - af(b))/(b - a)] = [(bf(a) - af(a))/(b - a)] \\ &= f(a) \end{aligned}$$

and

$$\begin{aligned} l(b) &= [(f(b) - f(a))/(b-a)]b + [(bf(a) - af(b))/(b-a)] \\ &= [(bf(b) - bf(a) + bf(a) - af(b))/(b-a)] = [(bf(b) - af(b))/(b-a)] \\ &= f(b) \end{aligned}$$

Thus, $l(a) = f(a)$, $l(b) = f(b)$ and $\partial/_{\partial R} f(a) = l'(c) = (f(b) - f(a))(b-a)$ for all c from (a, b) and all A-approximations of the point c. In addition, $g(a) = g(b) = 0$. By the Extended Rolle Theorem, there is a point $c \in [a, b]$ and an A-approximation R of c such that $\partial/_{\partial R}\, g(a) = 0$. Then by Corollary 4.2.4, we have

$$\partial/_{\partial R}\, g(a) = \partial/_{\partial R}\, (f - l)(a) = \partial/_{\partial R} f(a) - (f(b) - f(a))/(b-a) = 0$$

Consequently, $\partial/_{\partial R} f(a) = (f(b) - f(a))/(b-a)$.

Theorem is proved.

This gives us the classical result (Ross 1996; Burgin 2008a).

Corollary 4.2.7 (*Mean Value Theorem*) *If a function* $f(x)$ *is continuous on* [*a*, *b*], *differentiable in (a, b), and f(a) = f(b), then there is (at least one) point c* $\in (a, b)$ *such that* $\partial/_{\partial R} f(c) = (f(b) - f(a))/(b-a)$.

Remark 4.2.7 The condition that $f(x)$ is a continuous function is essential for validity of Theorem 4.2.9 because the Mean Value Theorem is not valid for arbitrary functions as the following example demonstrates.

Example 4.2.5 Let us take the function $f(x)$ that is equal to 1 at 1 and 0 at all other points. Then $f(1) - f(0)/(1 - 0) = 1$ for the interval [a, b] = [0, 1], while all derivatives $\partial/_{\partial R} f(c)$ are equal to 0 when c is an inner point of the interval [0, 1].

Let us define $Spec\, df(a) = \{r \in \boldsymbol{R}; \exists R \in appr(a)(\partial/_{\partial R} f(a) = r)\}$.

Corollary 4.2.8 *For a continuous real function f(x), Spec df(a) is a closed connected subset of* $\boldsymbol{R}$.

Corollary 4.2.9 *For any number* $r \in Spec\, \partial/_{\partial Q}\, g(a)$, *there is such an approximation* $R \in appr(a)$ that $\partial/_{\partial R}\, g(a) = r$.

Let a real function f be continuous at a point $a \in \boldsymbol{R}$.

Theorem 4.2.10 *For any fundamental (almost fundamental) one-sided partial derivative* $\partial/_{\partial R} f(a)$, *there is a two-sided derivative* $\partial/_{\partial Q} f(a)$ *such that* $\partial/_{\partial Q} f(a) = \partial/_{\partial R} f(a)$.

Proof We prove the statement only for right partial derivatives because for left partial derivatives the proof is the same. Lemma 4.2.2 demonstrates that it is possible to do this only for fundamental right partial derivatives $\partial/_{\partial R} f(a)$, that is, partial derivatives, which are built on such right A-approximations $R = \{<a_i, b_i>; i \in \omega\}$ for which $a_i = a$ in all intervals $<a_i, b_i>$ from R.

Let us consider a fundamental right A-approximations $R = \{<a, b_i>; i \in \omega\}$ and take such a sequence $\{d_i; i \in \omega\}$ that $\lim_{i\to\infty}(((f(b_i) - f(a)) \cdot (d_i - a))/$

$(b_i - a)^2) = 0$, $\lim_{i\to\infty}(f(d_i) - f(a))/(b_i - a) = 0$, $\lim_{i\to\infty}(d_i - a)/(b_i - a) = 0$ and $d_i < a$ for all $i \in \omega$. It is possible to do so because the function f is continuous at the point a.

Then for the approximation $Q = \{<d_i, b_i>; i \in \omega\}$, we have

$$\begin{aligned}\partial/_{\partial Q} f(a) &= \mathrm{Hn}((f(b_i) - f(d_i))/(b_i - d_i))_{i\in\omega} \\ &= \mathrm{Hn}((f(b_i) - f(a) + f(a) - f(d_i))/(b_i - d_i))_{i\in\omega} \\ &= \mathrm{Hn}((f(b_i) - f(a))/(b_i - d_i))_{i\in\omega} + \mathrm{Hn}((f(a) - f(d_i))/(b_i - d_i))_{i\in\omega} \\ &= \mathrm{Hn}((f(b_i) - f(a))/(b_i - d_i))_{i\in\omega}\end{aligned}$$

because $\mathrm{Hn}((f(b_i) - f(d_i))/(b_i - d_i))_{i\in\omega} = 0$ according to the choice of Q.

At the same time,

$$\begin{aligned}\partial/_{\partial R} f(a) &= \mathrm{Hn}((f(b_i) - f(a))/(b_i - a))_{i\in\omega} \\ &= \mathrm{Hn}((f(b_i) - f(a))/(b_i - d_i))_{i\in\omega} \cdot \mathrm{Hn}((b_i - d_i)/(b_i - a))_{i\in\omega} \\ &= \mathrm{Hn}((f(b_i) - f(a))/(b_i - d_i))_{i\in\omega}\end{aligned}$$

because $\mathrm{Hn}((b_i - d_i)/(b_i - a))_{i\in\omega} = \mathrm{Hn}((b_i - a + a - d_i)/(b_i - a))_{i\in\omega} = \mathrm{Hn}((b_i - a)/(b_i - a))_{i\in\omega} + \mathrm{Hn}((a - d_i)/(b_i - a))_{i\in\omega} = 1 + \mathrm{Hn}((a - d_i)/(b_i - a))_{i\in\omega}$ and $\lim_{i\to\infty}((a - d_i)/(b_i - a)) = 0$ by the choice of Q, i.e., $\mathrm{Hn}((a - d_i)/(b_i - a))_{i\in\omega} = 0$.

Thus, $\partial/_{\partial R} f(a) = \partial/_{\partial Q} f(a)$.

Theorem is proved.

Remark 4.2.6 The result of Theorem 4.2.10 remains true for all fundamental right (left) partial derivatives of f at a if the point $(a, f(a))$ is not isolated from the right and left in the graph $G = \{(z, f(z)); z \in X\}$ of f.

Remark 4.2.7 All conditions are essential for validity of Theorem 4.2.10.

Theorem 4.2.11 *If all one-sided almost fundamental (all right, left) partial derivatives $\partial/_{\partial R} f(a)$ are finite, then $f(x)$ is (right or respectively, left) continuous at the point a.*

Proof At first, let us assume that a function f is discontinuous, i.e., has a gap, from the right at a point a. It means that there is a sequence $\{x_i; i \in \omega\}$ such that $a < x_i$ for all $i \in \omega$ and $\lim_{i\to\infty} x_i = a$, but $\lim_{i\to\infty} f(x_i) \neq f(a)$. In this situation, it is possible to separate two cases: (1) the set $A = \{f(x_i); i \in \omega\}$ is bounded; (2) the set A is unbounded.

In the first case, the set A contains a convergent subsequence. By exclusion of some points from the sequence $\{x_i; i \in \omega\}$, we can obtain a sequence $\{f(x_j); i \in \omega\}$ such that it itself converges, i.e., there is a number $d = \lim_{j\to\infty} f(x_j)$ and $d \neq f(a)$. Then $|f(a) - f(x_j)| > |f(a) - d|/2$ for almost all $i \in \omega$. Taking the right A-approximation $R = \{<a, x_j>; i \in \omega\}$ of the point a, we see that the partial derivative $\partial/_{\partial R} f(a)$ is infinite. Indeed, $\partial/_{\partial R} f(a) = \mathrm{Hn}(\Delta_i^R f/\Delta_i^R x)$ where $\Delta_i^R f = f(a) - f(x_j)$ is always either larger or less than some fixed positive number, while $\Delta_i^R x = x_i - a \to 0$.

If the set A is unbounded, then it is possible to find a subsequence $\{x_j; j \in \omega\}$ of the sequence $\{x_i; i \in \omega\}$, such that the sequence $\{f(x_j); i \in \omega\}$ diverges. In this case, taking the right A-approximation $R = \{<a, x_j>; i \in \omega\}$ of the point a, we obtain the infinite partial derivative $\partial/_{\partial R} f(a)$. Thus, by the Law of Contraposition for propositions (cf. Church 1956), $f(x)$ is continuous from the right at the point a.

The proof for the left partial derivative is similar. The case when all one-sided almost fundamental partial derivatives are finite is implied by the two previous cases because f is continuous at the point a if and only if f is continuous at a from both sides.

Theorem is proved.

Corollary 4.2.10 *If all one-sided almost fundamental (right, left) partial derivatives $\partial/_{\partial R} f_x$ are finite for all x from X, then f is (right or respectively, left) continuous on X.*

Remark 4.2.8 The inverse of the statement of Theorem 4.2.11 is not true as the following example shows.

Example 4.2.6 Let us consider the following function

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ \sqrt{-x} & \text{if } x < 0 \end{cases}$$

We see that $f(x)$ is a continuous function on the whole real line $\boldsymbol{R}$. At the same time, all one-sided almost fundamental partial derivatives of $f(x)$ at the point 0 are infinite.

Let us take a real hypernumber α.

Proposition 4.2.2 *If $\partial/_{\partial R}\, g(a) = \alpha$ for some $R \in appr(a)$, then for any number $r \in Spec\,\alpha$, we have $\partial/_{\partial Q}\, g(a) = r$ for some $Q \in appr(a)$.*

Indeed, if $\partial/_{\partial R}\, g(a) = \alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ for some $R \in appr(a)$ and $r \in Spec\,\alpha$, then there is a subsequence $\boldsymbol{b} = (b_i)_{i\in\omega}$ of the sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ such that $r = \mathrm{Hn}(b_i)_{i\in\omega}$ (cf. Chap. 2). Taking a subapproximation Q of the A-approximation R that corresponds to the subsequence $\boldsymbol{b}$, i.e., elements of which have the same numbers in R as elements of $\boldsymbol{b}$ in $\boldsymbol{a}$, we see that $\partial/_{\partial Q}\, g(a) = \mathrm{Hn}(b_i)_{i\in\omega} = r$.

Let f be a real function and $\mathbf{T}$ is a set of A-approximations of a point a, i.e., $\mathbf{T} = \{R_\theta \in appr(a); \theta \in I\}$.

Definition 4.2.3 The set $\partial/_{\partial \mathbf{T}} f(a) = \{\partial/_{\partial R} f(a); R \in \mathbf{T}\}$ is called the *full* **T**-*derivative* of the function $f(x)$ at the point a.

Example 4.2.7 Let us consider the function $f(x) = |x|$. Then $\partial/_{\partial\, appr(a)} f(0) = [-1, 1]$, $\partial/_{\partial\, lappr\,(a)} f(0) = -1$, $\partial/_{\partial\, rappr\,(a)} f(0) = 1$, $\partial/_{\partial\, cappr\,(a)} f(0) = 0$, and $\partial/_{\partial\, bappr(a)} f(0) = [-1, 1]$.

Lemma 4.2.3 *If* **T** *and* **V** *are sets of* A-*approximations of a point a and* $\mathbf{T} \subseteq \mathbf{V}$, *then* $\partial/_{\partial \mathbf{T}} f(a) \subseteq \partial/_{\partial \mathbf{V}} f(a)$.

Proof is left as an exercise.

Theorem 4.2.6 implies the following result.

Corollary 4.2.11 *There is a real function f(x) and a real number a such that* $\partial/_{\partial\, appr\,(a)} f(a) = \boldsymbol{R}_\omega$.

Definition 4.2.4 A function f is called **T**-*differentiable* at a point a if for any R, $Q \in \mathbf{T}$, we have $\partial/_{\partial R} f(a) = \partial/_{\partial Q} f(a)$, i.e., the full **T**-derivative $\partial/_{\partial \mathbf{T}} f(a)$ has only one element.

Lemma 4.2.3 implies the following result.

Lemma 4.2.4 *If* **T** *and* **V** *are sets of* A-*approximations of a point a,* $\mathbf{T} \subseteq \mathbf{V}$ *and a function f is* **V**-*differentiable at a, then f is* **T**-*differentiable at a.*

Proof is left as an exercise.

Proposition 4.2.3 *If a set* ***T*** *of* A-*approximations of a point a is closed with respect to subapproximations and for some* A-*approximations R from the set* ***T***, $\partial/_{\partial R} f(a) = \alpha \notin \boldsymbol{R}$, *then the real function f is not* ***T***-*differentiable at a.*

Proof By Theorem 2.1.2, α is either an infinite hypernumber or a finite oscillating hypernumber. Let us assume that α is an infinite hypernumber. Then it is either an infinitely increasing or infinite positive or infinite expanding or infinitely decreasing hypernumber, and by Theorem 4.2.1, in the first three cases, for any hypernumber $\beta > \alpha$, there is a subapproximation Q of R such that $\partial/_{\partial Q} f(a) > \beta$. Thus, $\partial/_{\partial R} f(a) \neq \partial/_{\partial Q} f(a)$ and f is not **T**-differentiable at a because **T** is closed with respect to subapproximations.

When α is an infinitely decreasing hypernumber, then by Theorem 4.2.1, for any hypernumber $\beta < \alpha$, there is a subapproximation Q of R such that $\partial/_{\partial Q} f(a) < \beta$. Thus, $\partial/_{\partial R} f(a) \neq \partial/_{\partial Q} f(a)$ and f is not **T**-differentiable at a because **T** is closed with respect to subapproximations.

When $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$ is a finite hypernumber, then by Theorem 2.1.2, it is an oscillating hypernumber. So, by Proposition 2.1.7, *Spec* α contains, at least, two elements (say, u and v). Then Definitions 2.1.14 and 2.2.10, there is a subsequence $\boldsymbol{c} = (c_i)_{i\in\omega}$ of the sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ such that $\lim_{i\to\infty} c_i = u$. Consequently, $\partial/_{\partial R} f(a) \neq \partial/_{\partial Q} f(a)$ for the subapproximation Q of R that corresponds to the subsequence $\boldsymbol{c} = (c_i)_{i\in\omega}$, and f is not **T**-differentiable at a because **T** is closed with respect to subapproximations.

Proposition is proved.

To find connections with the classical calculus, we remind the definition of the classical derivative (Ross 1996; Stewart 2003). Let $X, Y \subseteq \boldsymbol{R}$, $f: X \to Y$ be a function, $a \in X$, $b \in \boldsymbol{R}$, and X contains some open interval $(a - k, a + k)$.

Definition 4.2.5

(a) A number b is called the *derivative* of $f(x)$ at a point $a \in X$ if $b = \lim_{i\to\infty}(f(a) - f(x_i))/(a - x_i)$ for all sequences $\{x_i; x_i \in (a-k, a+k);\ x_i \neq a; i = 1, 2, 3, \ldots\}$ converging to a.

(b) A function $f(x)$ is called *differentiable at a point* $a \in X$, if $f(x)$ has the derivative at a.

The derivative of the function $f(x)$ at a point a is denoted by $f'(a)$ or $\mathrm{d}/_{\mathrm{d}x}f(a)$.

It is known that if the derivative of $f(x)$ at a point a exists, then it is unique (Ross 1996).

Let us take the class $\mathrm{SA}_a = srappr(a) \bigcup slappr(a)$, which consists of all stable right and left A-approximations of a point a. Then Definitions 4.2.1 and 4.2.5 imply the following result.

Proposition 4.2.4 *The following conditions are equivalent:*

(a) A function $f(x)$ has the classical derivative $f'(a)$ at the point a.
(b) The function $f(x)$ is SA_a-differentiable at a.

Proposition 4.2.4 means that if a function has a derivative at a in the classical sense, then its right (left) sequential partial derivative at a is unique and coincides with the classical derivative of f at a. Moreover, if the right (left) partial derivative at a is unique and coincides with the left (right) partial derivative at a, then it coincides with the classical one and, as a consequence, the function that has a unique right partial derivative and left partial derivative, which coincide, is differentiable at a in the conventional sense.

Because global differentiation means differentiation at all points, Proposition 4.2.4 implies the following result.

Let $X \subseteq \boldsymbol{R}$.

Corollary 4.2.12 *The following conditions are equivalent:*

(a) A function $f(x)$ is SA_a-differentiable for all points from X
(b) The function $f(x)$ is differentiable in X

Corollary 4.2.12 means that if a function is differentiable in X in the classical sense, then all its partial derivatives at points of X are unique and coincide with the classical derivatives of f at points of X. Moreover, if all partial derivatives of a function f on X are unique, then they coincide with the classical derivatives of f in X and, as a consequence, the function f is differentiable in X in a conventional sense.

In some cases, taking a function f, we find that it has no derivative at a point a, but there is a right or/and left derivative at this point. For instance, the function $f(x) = |x|$ does not have a derivative at 0, but its right derivative at 0 is equal to 1, while its left derivative at 0 is equal to -1. It is known that a function $f(x)$ has the derivative at a point a if and only if it has both right and left derivatives at a and these derivatives coincide. That is why we also study relations between right and left sequential partial derivatives and right and left derivatives in the classical sense.

Let $X, Y \subseteq \boldsymbol{R}$, $f : X \to Y$ be a function, $a \in X$, $b \in \boldsymbol{R}$, and X contains some interval $[a, a + k)$ (interval $(a - k, a]$).

Definition 4.2.6

(a) A number b is called the *right (left) derivative* of $f(x)$ at a point $a \in X$ if $b = \lim_{i\to\infty}(f(a) - f(x_i))/(a - x_i)$ for all sequences $\{x_i; x_i \in (a, a+k);$ $\{i = 1, 2, 3, \ldots\}$ (correspondingly, all sequences $\{x_i; x_i \in (a-k, a); i = 1, 2, 3, \ldots\}$) converging to a.

(b) A function $f(x)$ is called *right (left) differentiable at a point* $a \in X$, if $f(x)$ has a right (left) derivative at a.

It is known that if the right or left derivative of $f(x)$ at a point a exists, then it is unique.

Definitions 4.2.1 and 4.2.6 directly imply the following result.

Proposition 4.2.5 *The following conditions are equivalent:*

(a) A function $f(x)$ has the right (left) derivative at the point a
(b) The function $f(x)$ is *srappr*(a)-differentiable (*slappr*(a)-differentiable) at a

Proposition 4.2.5 means that if a function has a right (left) derivative at a in the classical sense, then its right (left) partial derivative at a is unique and coincides with the classical right (left) derivative of f at a. Moreover, if the right (left) partial derivative at a is unique, then it coincides with the classical one and, as a consequence, the function that has a unique right (left) partial derivative is right (left) differentiable at a in a conventional sense.

Proposition 4.2.5 implies the following result.

Corollary 4.2.13 *The following conditions are equivalent:*

(a) A function $f(x)$ is *srappr*(a)-differentiable (*slappr*(a)-differentiable) for all points from X
(b) The function $f(x)$ is right (left) differentiable in X

Corollary 4.2.13 means that if a function is right (left) differentiable in X in the classical sense, then all its right (left) partial derivatives at points of X are unique and coincide with the classical right (left) derivatives of f at points of X. Moreover, if all right (left) partial derivatives of a function f on X are unique, then they coincide with the classical right (left) derivatives of f in X and, as a consequence, the function f is right (left) differentiable in X in a conventional sense.

In addition, if the conventional derivative exists, then it is equal to some sequential partial derivative of the same function at the same point. That is why all basic classical results for conventional derivatives follow from the corresponding results for sequential partial derivatives. For instance, Theorems 4.2.1–4.2.4 and Proposition 4.2.5 imply the following classical rules of differentiation, which are usually taught in all courses of calculus (cf. Ross 1996; Edwards and Penney 2002; Stewart 2003; Burgin 2008a).

Corollary 4.2.14 *For any differentiable at a point a real functions f and g, we have:*

(a) $(f+g)'(a) = f'(a) + g'(a)$
(b) $(f-g)'(a) = f'(a) - g'(a)$
(c) $(c \cdot f + d \cdot g)'(a) = c \cdot f'(a) + d \cdot g'(a)$

Corollary 4.2.15 (the *Product Rule*) *For any differentiable at a point a real functions f and g, we have*

$$(f \cdot g)'(a) = g(a) \cdot f'(a) + f(a) \cdot g'(a)$$

Corollary 4.2.16 (the *Quotient Rule*) *If functions f and g are differentiable a point a and g is not equal to* 0 *at a, then we have*

$$(f/g)'(a) = [g(a) \cdot f'(a) - f(a) \cdot g'(a)]/(g(a))^2$$

Corollary 4.2.17

(a) *Any right (left) differentiable at a point a real function f is right (left) continuous at a.*

(b) *Any differentiable at a point a real function f is continuous at a.*

Corollary 4.2.18

(a) *Any right (left) differentiable real function f is right (left) continuous.*

(b) *Any differentiable real function f is continuous.*

To conclude this chapter, it is necessary to remark that sequential partial derivatives are related not only to conventional derivatives but also to generalized derivatives in the sense of nonsmooth analysis (Clark 1983), weak derivatives, and distributional derivatives. For instance, it is possible to demonstrate that any distributional derivative is equal (in the sense of distributions) to some sequential partial derivative of an extended distribution (Burgin and Ralston 2004; Burgin 2010). This allows one to derive many properties of generalized derivatives, weak derivatives, and distributional derivatives from properties of sequential partial derivatives.

Chapter 5
How to Integrate Any Real Function

I'm not for integration and I'm not against it

(Richard Pryor 1940–2005)

In this chapter, we study problems of integration, demonstrating the advantages that transition to hypernumbers and extrafunctions gives for this field. It is well known that it is possible to integrate only some real functions. Shenitzer and Steprāns (1994) explain that while in the eyes of some mathematicians the Lebesgue integral was the final answer to the difficulties associated with integration and there is no *perfect* integral, there were others who were not willing to give up the search for the *perfect* integral, one which would make all functions integrable. Although efforts of different mathematicians extended the scope of integrable functions, their results only gave additional evidence for impossibility of such a perfect integral. In contrast to this, extending the concept of *integral*, which takes values in numbers, to the concept of a *partial hyperintegral*, which takes values in hypernumbers, it becomes possible to integrate any total real function. In particular, it is possible to define integrals that do not have a satisfactory definition in the conventional calculus, such as $\int_1^\infty (1/x)\,\mathrm{d}x$, $\int_1^\infty x^2\,\mathrm{d}x$ and $\int_1^\infty x\,\mathrm{d}x$.

It is possible to argue that in the conventional theory of integration, it is possible make the values of these and similar integrals equal to ∞. However, this brings us to contradictions if we assume that integration is a linear mapping (linear operator). Indeed, let us define

$$A = \int_1^\infty (1/x)\,\mathrm{d}x = \infty$$

$$C = \int_1^\infty x^2\,\mathrm{d}x = \infty$$

$$C = \int_1^\infty (1/x + 1/x^2)\,\mathrm{d}x = \infty$$

M. Burgin, *Hypernumbers and Extrafunctions: Extending the Classical Calculus*, SpringerBriefs in Mathematics, DOI 10.1007/978-1-4419-9875-0_5,

and

$$C = \int_1^{\infty} (-1/x + 1/x^2)\,\mathrm{d}x = -\infty$$

Then we have

$$B - A = \infty - \infty = 1$$

$$B - C = \infty - \infty = -\infty$$

$$A + C = \infty + \infty = \infty$$

and

$$B + D = \infty - \infty = 2$$

because taking $\int_1^{\infty} (1/x^2)\mathrm{d}x$ as an improper integral, we have $\int_1^{\infty} (1/x^2)\mathrm{d}x = \lim_{n\to\infty} \int_1^{n} (1/x^2)\mathrm{d}x = \lim_{n\to\infty}(-1/x|_1^n) = \lim_{n\to\infty}(-1/n - (-1)) = 1$

It means that such basic operations as addition and subtraction cannot be consistently defined in this situation because in one case $\infty - \infty = -\infty$, while in the other case $\infty - \infty = 2$. Besides, utilization of only two infinite values results in loss of information about integrals that take infinite values and consequently, about models of physical phenomena that use these integrals.

History of mathematics tells us that integration was introduced as an operation for measuring areas and volumes of geometrical figures (shapes). Later applications of integration were extended and integration has been used to measure and evaluate global characteristics of various objects, e.g., to find the length of a curve or the expectation of a random variable (random function). In doing this, it has been assumed that the value of these characteristics does not depend on the process of measurement.

In contrast to this, partial hyperintegration reflects a more refined situation. Namely, only in some cases, the result of a measurement theoretically does not depend on the process of this measurement, while in other cases different measurement techniques can give different results.

In Sect. 5.1, basic elements of the theory of covers and partitions are presented. Section 5.2 treats problems of hyperintegration of real functions over finite intervals, while Sect. 5.3 deals with hyperintegration of real functions over infinite intervals (spaces).

Various properties of partial hyperintegrals are obtained. Some of the properties are similar to properties of conventional integrals. For instance, the partial hyperintegral of the sum of two functions is equal to the sum of partial hyperintegral of each of these functions (Theorem 5.2.4). Other properties of partial hyperintegrals are essentially different from properties of any type of conventional integrals (either Riemann integral or Lebesgue integral or Stieltjes-Lebesgue or

Henstock-Kurzweil integral). For instance, the partial hyperintegral of any real function always exists (Theorem 5.2.1). However, as a rule, it is not unique and if we have a positive infinite partial hyperintegral of a real function *f* over an interval [*a*, *b*], then for any hypernumber α, there is a partial hyperintegral of the function *f* over the same interval such that its value is larger than α (Theorem 5.2.2).

It is necessary to remark that in this book, we consider only hyperintegrals of real functions, which are linear hyperfunctionals. More general hyperfunctionals are studied by Burgin (1991, 2004), while hyperintegration in more general spaces (in particular, in infinite dimensional spaces) is studied by Burgin (1995, 2004, 2005).

5.1 Covers and Partitions

Let X be a set.

Definition 5.1.1 A system $R = \{X_i; i \in I\}$ of sets X_i is called a *cover* of X if $X \subseteq \bigcup_{i \in I} X_i$.

Example 5.1.1 The system $R = \{[-n, n]; n \in \boldsymbol{N}\}$ of intervals $[-n, n]$ is a cover of the real line $\boldsymbol{R}$.

Example 5.1.2 The system $R = \{[n, n+1]; n \in \boldsymbol{Z}\}$ of intervals $[n, n+1]$ is a cover of the real line $\boldsymbol{R}$.

Lemma 5.1.1 *If* $R = \{X_i; i \in I\}$ *is a cover of* X *and* $Q \supseteq R$, *then* Q *is also a cover of* X.

Proof is left as an exercise.

Let us consider two covers $Q = \{Y_j; j \in J\}$ and $R = \{X_i; i \in I\}$ of a set X.

Definition 5.1.2
1. The cover Q is called a *refinement* of the cover R if for any element Y_j from Q, there is an element X_i from R such that $Y_j \subseteq X_i$.
2. The cover Q is said to be *finer* than the cover P.
3. The cover Q is called a *strong refinement* of the cover R if Q is a refinement of R and for any element X_i from R, there is a subset Q_i of the set Q that is a cover of X_i.
4. The refinement Q of R is called *finite* if each element X_i from R contains only a finite number of elements Y_j from Q.

We denote the relation "to be a refinement" by the symbol $\leq$, e.g., $Q \leq R$, and denote the relation "to be a strong refinement" by the symbol $\preccurlyeq$, e.g., $Q \preccurlyeq R$.

Example 5.1.3 The cover $R = \{[-2n, 2n]; n \in \boldsymbol{N}\}$ of the real line $\boldsymbol{R}$ is a refinement but not a strong refinement of the cover $Q = \{[-3n, 3n]; n \in \boldsymbol{N}\}$ of the real line $\boldsymbol{R}$.

Lemma 5.1.2 *Relation* $\preccurlyeq$ *is transitive and reflexive, i.e., it is a preorder in the set of all covers of a set* X.

Proof
(a) By Definition 5.1.2, $R \preccurlyeq R$ is true for any cover R, i.e., $\preccurlyeq$ is a reflexive relation.

(b) Let $R = \{X_i; i \in I\}$, $T = \{Z_k; k \in K\}$, and $Q = \{Y_j; j \in J\}$ are covers of the space X, $R \preccurlyeq Q$, and $Q \preccurlyeq T$. To show $R \preccurlyeq T$, we need to check conditions from Definition 5.1.2. The condition from Definition 5.1.2(1) is true because for any sets A, B, and C, inclusions $A \subseteq B$ and $B \subseteq C$ imply $A \subseteq C$.

Taking an element Z_k from T, by Definition 5.1.2, we have a subset $Q_k = \{Y_{ki}; i \in I\}$ of the set Q that is a cover of Z_k, i.e., $Z_k = \bigcup_{i \in I} Y_{ki}$. In addition, by Definition 5.1.2, for each element Y_{ki} from Q_k, there is a subset $R_{ki} = \{X_{kij}; j \in J\}$ of the set R that is a cover of Y_{ki}, i.e., $Y_{ki} = \bigcup_{j \in J} X_{kij}$. Thus, for any element Z_k from T, we have a subset $R_k = \bigcup_{i \in I} R_{ki}$ of the set R that is a cover of Z_k because $Z_k = \bigcup_{i,j} X_{kij}$. This gives us the condition from Definition 5.1.2(3), and consequently, shows that $\preccurlyeq$ is a transitive relation.

Lemma is proved.

In a general case, relation $\preccurlyeq$ is not a partial order (cf. Appendix). To show that relations $R \preccurlyeq Q$ and $Q \preccurlyeq R$ do not imply the relation $R = Q$ is a general case, we consider the following example.

Example 5.1.4 Let us consider the unit interval $E = [0, 1]$ and the following system of its subsets:

$$R_1 = \left\{\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right]\right\}, \quad Q_1 = \left\{\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right]\right\}$$

$$R_2 = \left\{\left[0, \frac{1}{8}\right], \left[\frac{1}{8}, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{3}{8}\right], \left[\frac{3}{8}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{5}{8}\right], \left[\frac{5}{8}, \frac{3}{4}\right], \left[\frac{3}{4}, \frac{7}{8}\right], \left[\frac{7}{8}, 1\right]\right\}, \ldots,$$

$$R_n = \{[0, (1/2^{2n-1})], [(1/2^{2n-1}), (1/2^{2n-2})], [(1/2^{2n-2}), (1/2^{2n-3})], \ldots, [((2^{2n-1} - 1)/(2^{2n-1})), 1]\}$$

$$Q_n = \{[0, (1/2^{2n})], [(1/2^{2n}), (1/2^{2n-1})], \ldots, [((2^{2n} - 1)/(2^{2n})), 1]\}, \ldots$$

By Definition 5.1.2, we have

$$\cdots Q_n \leq R_n \leq Q_{n-1} \leq \cdots \leq R_3 \leq Q_2 \leq R_2 \leq Q_1 \leq R_1$$

We define $R = \bigcup_{i=1}^{\infty} R_i\, R_i$ and $Q = \bigcup_{i=1}^{\infty} Q_i$. As each R_i and each Q_i is a cover of the interval E, R, and Q are also covers of E. By Definition 5.1.2, we have $R \preccurlyeq Q$, and $Q \preccurlyeq R$, but $R \neq Q$.

Definition 5.1.3 The cover Q is called a *subcover* of the cover R if Q is a subset of R.

We denote the relation "to be a subcover" by the symbol $\subseteq$, e.g., $Q \subseteq R$.

Example 5.1.5 The system $Q = \{[-2n, 2n]; n \in N\}$ is a subcover of the cover $R = \{[-n, n]; n \in N\}$ of the real line R.

As relation $\subseteq$ is a partial order in the domain of sets, we have the following result.

Lemma 5.1.3 *Relation* $\subseteq$ *is a partial order in the set of all covers of a set* X.

Note that if Q is a subcover of R, then Q is a refinement of R.
There are different kinds of covers.

Definition 5.1.4 A cover $R = \{X_i; i \in I\}$ of the space X is called:

(a) *Finite* if the set I is finite
(b) *Cumulative* if $I = \mathbf{N}$ and $X_i \subseteq X_{i+1}$ for all $i = 1, 2, 3, \ldots$
(c) *Reducible* if there is a set X_i from R such that $Q = R \backslash \{X_i\}$ is also a cover of the space X
(d) An *interval cover* if $I = \{1, 2, 3, \ldots\}$ and $X_i = [a_i, b_i]$ with $a_i \leq b_i$ for all $i = 1, 2, 3, \ldots$
(e) An *interval generated cover* if $I = \mathbf{N}$ and X_i is a finite union of intervals $[a_{ij}, b_{ij}]$ with $a_{ij} \leq b_{ij}$, $j \in J_i$, for all $i = 1, 2, 3, \ldots$
(f) A *linear interval cover* if $I = \mathbf{N}$ and $X_i = [a_i, b_i]$ with $a_i \leq b_i$ and $a_i < a_{i+1}$ for all $i = 1, 2, 3, \ldots$
(g) *Internal* if $X_i \subseteq X$ for all $i \in I$

Here we mostly consider internal interval covers.

Proposition 5.1.1
(a) *If* $R = \{X_i; i \in I\}$ *is a (cumulative or/and finite or/and internal) cover of* X *and* $Q = \{Y_i; i \in I\}$ *is a (cumulative or/and finite or/and internal) cover of* Y*, then* $\{X_i \bigcup Y_i; i \in I\}$ *is a (cumulative or/and finite or/and internal) cover of* $X \bigcup Y$.
(b) *If* $R = \{X_i; i \in I\}$ *is a (finite) cover of* X *and* $Q = \{Y_j; j \in J\}$ *is a (finite) cover of* Y*, then* $R \bigcup Q = \{X_i, i \in I; Y_j; j \in J\}$ *is a (finite) cover of* $X \bigcup Y$.

Indeed, if R covers X and Q covers Y, then $\{X_i \bigcup Y_i; i \in I\}$ covers $X \bigcup Y$. The inheritance of the corresponding properties also directly follows from definitions. Besides, as $X \subseteq \bigcup_{i \in I} X_i$ and $Y \subseteq \bigcup_{j \in J} Y_j$, we have

$$X \bigcup Y \subseteq \left(\bigcup\nolimits_{i \in I} X_i\right) \cup \left(\bigcup\nolimits_{j \in J} Y_j\right)$$

Example 5.1.6 Let us consider the unit interval $E = (0, 1)$ and the following system of intervals:

$$R_1 = \left[\frac{1}{4}, \frac{3}{4}\right], R_2 = \left[\frac{1}{8}, \frac{7}{8}\right], \ldots, R_n = [1/2^{2n+1}, (2^{2n+1} - 1)/(2^{2n+1})], \ldots$$

The system $R = \{R_i;\ i = 1, 2, 3, \ldots\}$ is a cumulative cover of E.

The cover R from Example 5.1.1 is cumulative and reducible, while the cover R from Example 5.1.2 is irreducible and not cumulative.

Lemma 5.1.4 $R \bigcup Q$ *is a finite cover if and only if both* R *and* Q *are finite covers.*

Proof is left as an exercise.

Lemma 5.1.5 *Any cumulative cover is reducible.*

Indeed, if we eliminate any element from an infinite cumulative cover or the first element from a finite cumulative cover, then the union of all other sets will still cover the whole initial space.

Lemma 5.1.6 *An interval cumulative cover of a finite closed interval* $[a, b]$ *contains this interval as its element.*

Indeed, if $R = \{X_i; i = 1, 2, 3, \ldots\}$ is a cumulative cover of the interval $[a, b]$, then there is a number i such that $a \in X_i$ and there is a number j such that $b \in X_j$. Then $[a, b] \subseteq X_k$ where $k = \max\{i, j\}$ because X_k is an interval.

Lemma 5.1.7 *A subcover of a cumulative cover of X is a cumulative cover of X.*

Proof is left as an exercise.

Lemma 5.1.8 *An irreducible cover does not have proper subcovers.*

Proof is left as an exercise.

Proposition 5.1.2 *If R is an infinite cumulative cover of a space X, then any infinite subset of R is a cumulative cover of the same space X.*

Indeed, if R is an infinite cumulative cover of a space X and U is an infinite subset of R, then any element X_i from R is a subset of some element X_j from U. Thus, U is a cover of X, and by Lemma 5.1.7, it is a cumulative cover of X.

An interval cumulative cover of an interval in $\boldsymbol{R}$ has the form $R = \{[a_i, b_i]; a_i \leq b_i, i = 1, 2, 3, \ldots\}$ and $a_{i+1} \leq a_i < b_i \leq b_{i+1}$ for all $i = 1, 2, 3, \ldots$.

Let X be a topological space.

Definition 5.1.5 A *partition* of X is an internal cover $R = \{X_i; i \in I\}$ of X such that the intersection of any two sets from R is a subset of the boundaries of these sets.

Note that if X is a set, i.e., a topological space with the discrete topology, then a partition of X is a cover of X that consists of disjoint subsets of X. This coincides with the concept of conventional partitions of sets (Abian 1965).

Note that for any partition $R = \{[a_i, b_i]; a_i \leq b_i, i \in I\}$ of X, it is possible to change the indexing so that $b_i \leq a_j$ if $i < j$. Thus, we assume that this condition is always true in what follows.

If X is a subset of $\boldsymbol{R}$, then an interval cover $R = \{[a_i, b_i]; a_i \leq b_i, i \in I\}$ of X is a partition of X if for all $i \in I, a_{i+1} = b_i$ or intervals $[a_i, b_i]$ and $[a_{i+1}, b_{i+1}]$ do not have common points.

The cover in Example 5.1.2 is a partition, while the cover in Example 5.1.1 is not a partition. Each cover R_i and each cover Q_i in Example 5.1.3 is a partition, while the cover in Example 5.1.4 and the covers R and Q from Example 5.1.3 are not partitions.

In what follows, we consider only the real line $\boldsymbol{R}$ and intervals in it as the space X and mostly *interval partitions*, i.e., partitions elements of which are closed intervals of the form $[a, b]$ with $a, b \in \boldsymbol{R}$. So, in this chapter, the term *partition* means an interval partition of the form $R = \{[a_i, b_i]; a_i \leq b_i, i \in I\}$.

Lemma 5.1.9 *An interval cover* $R = \{[a_i, b_i]; a_i \leq b_i, i \in I\}$ *of a finite or infinite interval in the real line* $\boldsymbol{R}$ *is a partition if and only if for all* $i \in I$, *we have* $a_{i+1} = b_i$.

Proof is left as an exercise.

There are different kinds of partitions.

Definition 5.1.6
A partition $R = \{[a_i, b_i]; a_i \leq b_i, i \in I\}$ of X is called:

(a) *Finite* if the set I is finite
(b) *Reducible* if there is a set X_i from R such that $Q = R \backslash \{X_i\}$ is also a partition of the space X
(c) *Uniform* if $[a_i, b_i] = [a_{i+1}, b_{i+1}]$ for all $i \in I$
(d) *Regular* if $a_i < b_i$ for all $i \in I$
(e) *Singular* if $a_i = b_i$ at least for one $i \in I$

It means that some elements of a singular partition are points, while regular partitions do not have points as their elements.

Lemma 5.1.9 shows that it is possible to represent any finite regular partition $R = \{[a_i, b_i]; a_i \leq b_i, i = 0, 1, 2, 3, \ldots, n-1\}$ of a finite interval $[a, b]$ by the sequence of points

$$a = a_0 < a_1 < a_2 < \cdots < a_n = b$$

where $a_{i+1} = b_i$ ($i = 0, 1, 2, 3, \ldots, n-1$). In other words, the interval $[a, b]$ is divided into n subintervals $[a_i, a_{i+1}]$ by choosing points $a = a_0 < a_1 < a_2 < \cdots < a_i < a_{i+1} < \cdots < a_n = b$. This gives a *dotted representation* of R and all these points are called *points of the partition R*.

The same is true for infinite regular partitions and infinite intervals, i.e., it is possible to represent any regular partition by a sequence of points that are ends of the partition intervals. Indeed, if $R = \{[a_i, b_i]; a_i \leq b_i, i \in I\}$ is a regular partition of $\boldsymbol{R}$, we can take $I = \boldsymbol{Z}$. Then this partition is uniquely represented by the sequence $\{c_i;\ i = 1, 2, 3, \ldots\}$ where $c_1 = a_0$, $c_{2k-1} = a_{-k}$, $c_{2k} = a_k$, and $k = 1, 2, 3, \ldots$. For infinite intervals (a, ∞) and $(-\infty, a)$ with $a \in \boldsymbol{R}$, the construction is similar. For the infinite interval $[a, \infty)$ with $a \in \boldsymbol{R}$, it is possible to enumerate intervals from a partition $R = \{[a_i, b_i]; a_i \leq b_i, i = 0, 1, 2, 3, \ldots\}$ of $[a, \infty)$ so that $a_0 = a$, $a_1 = b_0$, $a_2 = b_1$, and so on. With such an enumeration, R is uniquely represented by the sequence $a_0 < a_1 < a_2 < \cdots$. For the infinite interval $(-\infty, a]$ with $a \in \boldsymbol{R}$, it is possible to enumerate intervals from a partition $Q = \{[c_i, d_i]; c_i \leq d_i,\ i = 0, 1, 2, 3, \ldots\}$ of $(-\infty, a]$ so that $d_0 = a, d_1 = c_0, d_2 = c_1$, and so on. With such an enumeration, Q is uniquely represented by the sequence $d_0 > d_1 > d_2 > \cdots$.

In such a way, we obtain a dotted representation of a regular partition, which is frequently used by mathematicians. However, a partition consists of intervals and its points give only a representation of this partition.

Lemma 5.1.10 *Any regular partition is irreducible.*

Indeed, if we eliminate any interval $[a_i, b_i]$ from the partition, then the remaining interval do not cover the whole space X.

Definition 5.1.7 The *norm* (or *mesh*) $||R||$ of a partition $R = \{[a_i, b_i]; a_i \leq b_i, i \in I\}$ is the supremum of lengths of the intervals $[a_i, b_i]$, namely,

$$||R|| = \sup\{|b_i - a_i|; i \in I\}$$

For a finite partition $P = \{[a_i, b_i]; i = 1, 2, 3, \ldots, n\}$, the norm is the length of the longest of these subintervals, i.e., we have

$$||R|| = \max\{|b_i - a_i|; i = 1, ..., n\}$$

It is easy to see that any uniform partition is uniformly bounded.

Lemma 5.1.11 *Any finite partition* R *of an infinite interval has an infinite norm.*

Indeed, a finite number of finite intervals can cover only a finite interval.

Definition 5.1.8
A partition $R = \{[a_i, b_i]; a_i \leq b_i, i \in I\}$ of X *is called:*

(a) δ-*uniformly* bounded *if* $||R|| < \delta$ *for a positive real number* δ.
(b) *Uniformly* bounded *if there is a real number* δ *such that* $||R|| < \delta$.

Lemma 5.1.12 *Any partition R of a finite interval is uniformly bounded.*

Indeed, it is bounded by the length of the whole interval.

Corollary 5.1.1 *Any partition R of an interval [a, b] is* δ*-uniformly bounded where* $\delta = |b - a|$.

Definition 5.1.9
(a) The partition $Q = \{[c_j, d_j]; c_j \leq d_j, j \in J\}$ is called a *refinement* of a partition $R = \{[a_i, b_i]; a_i \leq b_i, i \in I\}$ if Q is a refinement of R as a cover.
(b) The partition Q is said to be *finer* than the partition P.

We denote the relation "*to be a refinement of a partition*" by the symbol $\precsim$, e.g., $Q \precsim P$.

Proposition 5.1.3 *Any finite refinement of an interval partition is a strong refinement.*

Proof Let us consider a partition $R = \{[a_i, b_i]; a_i \leq b_i, i \in I\}$ of either an interval $[a_i, b_i]$ or real line $\boldsymbol{R}$ or a ray in this line and its finite refinement $Q = \{[c_j, d_j]; c_j \leq d_j, j \in J\}$. As any two intervals from R either are disjoint or have only one common point, each interval from Q belongs exactly one interval $[a_i, b_i]$ from R due to Definition 5.1.8.a. As Q is a cover of X, all intervals from Q that are subsets of the interval $[a_i, b_i]$ cover this interval because the union of these intervals from Q is a closed set as the union of a finite number of closed sets

(Kelly 1957) and this union contains the open interval (a_i, b_i). Thus, by Definition 5.1.2, Q is a strong refinement of R.

Proposition is proved.

Remark 5.1.1 For infinite refinements, the statement of Proposition 5.1.3 is not true as the following example demonstrates.

Example 5.1.7 Let us consider the interval $X = [0, 2]$ and two its partitions $R = \{[0, 1], [1, 2]\}$ and $Q = \{[0, 1/2], [1/2, 3/4], [3/4, 7/8], \ldots, [((2^{2n} - 1)/(2^{2n})), ((2^{2(n+1)} - 1)/(2^{2(n+1)}))]\}, \ldots, [1, 2]\}$. By definition, Q is a refinement of R. However, it is neither a finite refinement of R as the interval [0, 1] contains infinitely many elements from Q nor a strong refinement as the interval [0, 1] is not covered by elements from Q.

Proposition 5.1.3 implies the following result.

Lemma 5.1.13 *A partition P is a refinement of a partition Q of the same interval if and only if the dotted representation of P contains all points of the dotted representation of Q and possibly some other points as well.*

In contrast to $\leqslant$, relation $\lesssim$ is antisymmetric.

Lemma 5.1.14 *If* P *and* Q *are partitions, $R \lesssim Q$ and $Q \lesssim R$, then $Q = R$.*

Proof Let us take two partitions $R = \{X_i; i \in I\}$ and $Q = \{Y_j; j \in J\}$ of the space X, such that $R \lesssim Q$ and $Q \lesssim R$. Then, $Q \lesssim R$ implies $\forall \mathrm{Y_j} \in \mathrm{Q} \exists \mathrm{X_i} \in \mathrm{R}$ such that $Y_j \subseteq X_i$ while $R \lesssim Q$ implies that each $X_i \subseteq Y_k$ for some $Y_k \in Q$.

As Q is a partition, the inclusion $Y_j \subseteq X_i \subseteq Y_k$ implies $Y_j = Y_k$ and $j = k$. Consequently, for any $Y_j \in Q$, there is $X_i \in R$ such that $Y_j = X_i$.

Symmetry of the initial conditions implies that for any $X_i \in R$, there is $Y_j \in Q$ such that $Y_j = X_i$. As all Y_j from Q are different from one another and all X_i from R are different from one another, we have $R = Q$.

Lemma is proved.

Note that covers R and Q in Example 5.1.4 are not partitions.

Proposition 5.1.4 *Relation $\lesssim$ is a partial order in the set of all partitions of an interval.*

Proof It is necessary to show that $\lesssim$ is a reflexive transitive antisymmetric relation. By Lemma 5.1.14, $\lesssim$ is an antisymmetric relation and by Lemma 5.1.2, relation $\lesssim$ is transitive and reflexive.

Proposition is proved.

Proposition 5.1.5 *If R is a partition (interval partition) of X, Q is a partition (interval partition) of Y, and X and Y do not intersect or have only one common point, then $R \bigcup Q$ is a partition (interval partition) of $X \bigcup Y$.*

Proof directly follows from Definition 5.1.5.

Corollary 5.1.2 *If R is a partition (interval partition) of an interval* $[a, b]$, *Q is a partition (interval partition) of an interval [b, c], then $R \bigcup Q$ is a partition (interval partition) of the interval [a, c] and* $||R \bigcup Q|| = \max\{||R||, ||Q||\}$.

Note that if R and Q are partitions of the same space X, then $R \bigcup Q$ is a partition of X if and only if $R = Q$.

Lemma 5.1.15 *If Q is a refinement of a partition R, then $||Q|| \leq ||R||$.*

Indeed, all intervals from Q are subintervals of intervals from R and the mesh of a subinterval is less than the mesh of the interval.

Lemma 5.1.16 *Any two disjoint partitions P and Q have a common refinement.*

Indeed, taking all points from the dotted representations of P and Q, re-numbered in order, we obtain the dotted representation of their common refinement. This new partition consists of all interval intersections of the intervals from both partitions, i.e., if $R = \{[a_i, b_i]; i \in I\}$ and $Q = \{[c_j, d_j]; j \in J\}$, then $P \vee Q = \{[p_t, q_t]; [p_t, q_t] = [a_i, b_i] \cap [c_j, d_j]; [a_i, b_i] \in R, [c_j, d_j] \in Q\}$. In such a way, we build the largest *common refinement*, i.e., the largest partition that is a refinement of both P and Q, is denoted by $P \vee Q$.

Let us take a subset X of $\boldsymbol{R}$ and a function $f : \boldsymbol{R} \to \boldsymbol{R}$.

Definition 5.1.10

1. A system $R = \{(X_i, x_i); x_i \in X_i \subseteq X, i \in I\}$ is called a *tagged cover* (*partition*) of X if $^{-}R = \{X_i; i \in I\}$ is a cover (partition) of X, which is called the *untagging* of R.
2. A system $R = \{(X_i, d_i); x_i, d_i \in \boldsymbol{R}, i \in I\}$ is called a *labeled cover* (*partition*) of X if $^{-}R = \{X_i; i \in I\}$ is a cover (partition) of X.
3. A system $R = \{(X_i, f(x_i)); x_i \in X_i \subseteq X, i \in I\}$ is called a *cover* (*partition*) of X *labeled by* the function f if $R^\circ = \{(X_i, x_i); x_i \in X_i \subseteq X, i \in I\}$ is a tagged cover (partition) of X, which is called the *tagging* of R.

Example 5.1.8 The system $R = \{([-n, n], n); n \in \boldsymbol{N}\}$ is a tagged cover of the real line $\boldsymbol{R}$.

Example 5.1.9 The system $R = \{([n, n+1], n); n \in \boldsymbol{Z}\}$ is a tagged partition of the real line $\boldsymbol{R}$.

Example 5.1.10 The system $R = \{([-2n, 2n], n^2); n \in \boldsymbol{N}\}$ is a cover of the real line $\boldsymbol{R}$ labeled by the function $f(x) = x^2$.

Example 5.1.11 The system $R = \{([-2n, 2n], 0); n \in \boldsymbol{N}\}$ is a cover of the real line $\boldsymbol{R}$ labeled by the function $f(x) = 0$ and at the same time, a tagged cover of $\boldsymbol{R}$.

Relation and other constructions introduced for covers and partitions induce corresponding relation and constructions of tagged covers and partitions. For instance, we have the following concepts.

Definition 5.1.11 The *norm* (*mesh*) $||R||$ of a tagged partition R is the norm (mesh) $||R||$ of the partition $\underline{R}$, i.e., $||R|| = ||\underline{R}||$.

Definition 5.1.12 A *tagged interval partition* (*cover*) is an interval partition R together with a distinguished point of every interval from R.

The systems R in Examples 5.1.8, 5.1.10, and 5.1.11 are tagged interval covers, while the cover in Example 4.1.9 is a tagged interval partition.

Definition 5.1.13 In a *right* (*left*) *tagged interval partition* R the right (left) end of each interval from R is taken as the distinguished point of this interval.

The system R in Example 4.1.9 is a left tagged interval partition.

Definition 5.1.14 An arbitrary function $\delta : [a, b] \to \boldsymbol{R}^{++}$ is called a *gauge*.

This is the basic concept for the Henstock-Kurzweil or gauge integration (cf. Bartle 2001).

Definition 5.1.15

(a) A tagged partition Q is a *refinement* of a tagged partition P if the partition $\underline{Q}$ is a refinement of the partition $\underline{P}$.
(b) A tagged interval partition $Q = \{([c_j, d_j], s_j); j \in J\}$ is a *robust refinement* of a tagged partition $P = \{([a_i, b_i], t_i); i \in I\}$ if the partition $\underline{Q}$ is a refinement of the partition $\underline{P}$ and when an interval $[c_j, d_j]$ from Q contains a tag t_i from P, then this is the tag in Q of the interval $[c_j, d_j]$, i.e., $t_i = s_j$.

As before, we denote the relation "to be a refinement" for tagged partition by the symbol $\preccurlyeq$ e.g., $Q \preccurlyeq P$ and the relation "to be a robust refinement" by the symbol $\ll$, e.g., $Q \ll P$.

Many properties of these constructions remain the same.

Proposition 5.1.6 *If* $R = \{X_i; i \in I\}$ *is a (cumulative or/and finite or/and internal) cover of* X *and* $Q = \{Y_j; j \in J\}$ *is a (cumulative or/and finite or/and internal) cover of* Y, *then* $R \bigcup Q$ *is a (cumulative or/and finite or/and internal) cover of* $X \bigcup Y$.

Proof is based on Proposition 5.1.1.

Proposition 5.1.4 implies the following result.

Proposition 5.1.7 *Relation* $\preccurlyeq$ *is a preorder in the set of all partitions of a given interval.*

Lemma 5.1.17 *Relation* $\ll$ *implies relation* $\preccurlyeq$.

Proof is left as an exercise.

Proposition 5.1.8 *Relation* $\ll$ *is a partial order in the set of all partitions of an interval.*

Proof is left as an exercise.

Proposition 5.1.9 *If a right (left) tagged interval partition* Q *is a robust refinement of a tagged interval partition* P, *then* P *is also right (left) tagged interval partition.*

Indeed, in a robust refinement, all tags of P are also tags of Q.

Definition 5.1.16

(a) A function

$$\delta : [a, b] \to \boldsymbol{R}^{++}$$

is called a *gauge* of a tagged interval partition $P = \{([a_i, b_i], c_i); i = 1, 2, 3, \ldots, n\}$ if for all $i = 1, 2, 3, \ldots, n$, we have

$$|b_i - a_i| < \delta(c_i)$$

(b) In this case, the partition P is called δ-*fine*.

The following result was proved in the theory of gauge integration.

Theorem 5.1.1 (Fineness Theorem *also called* Cousin's lemma) *For any interval* $[a, b]$ *with* $a < b$ *and any gauge* $\delta(x)$*, there is a* δ*-fine tagged partition of* $[a, b]$.

Proof For any x from $[a, b]$, we define $I_x = [x - \delta(x), x + \delta(x)]$. In addition, we define $I_a = [a, a + \delta(a)]$ and $I_b = [b - \delta(b), b]$. The system $C = \{I_x; x \in [a, b]\}$ is a cover of $[a, b]$. As $[a, b]$ is a compact space, there is a finite cover $D = \{I_{x_1}, I_{x_2}, \ldots, I_{x_n}; I_{x_i} \in C, i = 1, 2, 3, \ldots, n\}$.

It is possible to assume that D is a minimal cover of $[a, b]$ because if it is not minimal, it is possible to exclude some of the intervals I_{x_i} from D to get a minimal cover. It is possible to assume that $x_1 < x_2 < \cdots < x_n$. As D is a minimal cover of $[a, b]$, we have $x_{i+1} - \delta(x_{i+1}) \leq x_i + \delta(x_i)$ for all $i = 1, 2, 3, \ldots, n$.

If $x_i \leq x_{i+1} - \delta(x_{i+1}) \leq x_i + \delta(x_i) \leq x_{i+1}$, then we define $a_i = 1/2(x_i + \delta(x_i) - x_{i+1} + \delta(x_{i+1}))$, i.e., a_i is the midpoint of the interval $[x_{i+1} - \delta(x_{i+1}), x_i + \delta(x_i)]$.

If $x_{i+1} \leq x_i + \delta(x_i)$, then we take x_{i+1} instead of $x_{i+1} - \delta(x_{i+1})$ for building $a_i = 1/2(x_i + \delta(x_i) - x_{i+1})$, i.e., a_i is the midpoint of the interval $[x_{i+1}, x_i + \delta(x_i)]$.

If $x_{i+1} - \delta(x_{i+1}) \leq x_i$, then we take x_i instead of $x_i + \delta(x_i)$ for building $a_i = 1/2(x_i + \delta(x_i) - x_{i+1})$, i.e., a_i is the midpoint of the interval $[x_{i+1} - \delta(x_{i+1}), x_i]$.

This gives us the tagged partition $P = \{([a_0 = a, a_1], x_1), ([a_1, a_2], x_2), \ldots, ([a_{n-1}, a_n = b]), x_n)\}$. As each x_i belongs to $[a_{i-1}, a_i]$ and $|a_i - a_{i-1}| < \delta(x_i)$ for all $i = 1, 2, 3, \ldots n$, this tagged partition is δ-fine.

Theorem is proved.

For integration, we need not only separate partitions, but sequences of partitions.

Definition 5.1.17
A sequence $\mathbf{L} = \{P_i; i = 1, 2, 3, \ldots\}$ of partitions P_i is called:

(a) *Decreasing* if the inequality $||P_i|| > ||P_{i+1}||$ for norms is true for all $i = 1, 2, 3, \ldots$.
(b) *Converging* if $\lim_{i\to\infty} ||P_i|| = 0$.
(c) *Refined* if P_{i+1} is a refinement of P_i for all $i = 1, 2, 3, \ldots$.
(d) *Expanding* if there is a system of intervals $R = \{[a_i, b_i]; i \in I\}$ such that $a_{i+1} \leq a_i \leq b_i \leq b_{i+1}$ and P_i is a partition of the interval $[a_i, b_i]$ for all $i = 1, 2, 3, \ldots$.
(e) *Covering* $\boldsymbol{R}$ if it is expanding and the corresponding system of intervals $R = \{[a_i, b_i]; i \in I\}$ is a cumulative interval cover of $\boldsymbol{R}$.

Note that P_i can be partitions of different intervals.

Proposition 5.1.6 implies the following result.

Proposition 5.1.10 *If* $\mathbf{L} = \{P_i; i = 1, 2, 3, \ldots\}$ *is a sequence of (interval and/or converging and/or refined) partitions* P_i *of a space X,* $\mathbf{K} = \{R_i; i = 1, 2, 3, \ldots\}$ *is a sequence of (interval and/or converging and/or refined) partitions* R_i *of a space Y and X and Y do not intersect or have only one common point, then* $\mathbf{L} \bigcup \mathbf{K} = \{P_i \bigcup R_i; i = 1, 2, 3, \ldots\}$ *is a partition (interval partition and/or converging partition and/or refined partition) of* $X \bigcup Y$.

Corollary 5.1.3 *If* $\mathbf{L} = \{P_i; i = 1, 2, 3, \ldots\}$ *is a sequence of (interval and/or converging and/or refined) partitions* P_i *of an interval* $[a, b]$ *and* $\mathbf{K} = \{R_i; i = 1, 2, 3, \ldots\}$ *is a sequence of (interval and/or converging and/or refined) partitions* R_i *of an interval* $[b, c]$*, then* $\mathbf{L} \bigcup \mathbf{K} = \{P_i \bigcup R_i; i = 1, 2, 3, \ldots\}$ *is a partition (interval partition and/ or converging partition and/or refined partition, correspondingly) of the interval* $[a, c]$.

Proposition 5.1.11 *A subsequence of a decreasing (converging and/or refined) sequence of partitions is also a decreasing (converging and/or refined) sequence of partitions.*

Proof Let us take a decreasing sequence $\mathbf{L} = \{P_i; i = 1, 2, 3, \ldots\}$ of partitions and its subsequence $\mathbf{K} = \{R_i; i = 1, 2, 3, \ldots\}$. Then for any partitions R_i and R_{i+1} from $\mathbf{K}$, there are partitions P_l and P_j from $\mathbf{K}$, such that $l < j$, $R_i = P_l$, and $R_{i+1} = P_j$. As the relation $<$ is transitive, we have $||P_l|| = ||R_i|| > ||R_{i+1}|| = ||P_j||$. As i is an arbitrary natural number, the sequence $\mathbf{K}$ is decreasing.

Any subsequence of a converging sequence of real numbers is converging and has the same limit (Ross 1996; Burgin 2008a). Consequently, any subsequence of a converging sequence of partitions is also a converging sequence of partitions.

Let us take a refined sequence $\mathbf{L} = \{P_i; i = 1, 2, 3, \ldots\}$ of partitions and its subsequence $\mathbf{K} = \{R_i; i = 1, 2, 3, \ldots\}$. Then for any partitions R_i and R_{i+1} from $\mathbf{K}$, there are partitions P_l and P_j from $\mathbf{K}$, such that $l<j$, $R_i=P_l$, and $R_{i+1} = P_j$. As by Proposition 5.1.4, the relation $\lesssim$ is transitive, we have $R_{i+1} = P_j \lesssim P_l = R_i$. As i is an arbitrary natural number, the sequence $\mathbf{K}$ is refined.

Proposition is proved.

Definition 5.1.18 A sequence $\mathbf{L} = \{P_i; i = 1, 2, 3, \ldots\}$ of tagged partitions P_i is called:

(a) *Decreasing* if the inequality $||P_i|| \geq ||P_{i+1}||$ for norms is true for all i = 1, 2, 3, ...
(b) *Converging* if $\lim_{i \to \infty} ||P_i|| = 0$.
(c) *Refined* if P_{i+1} is a robust refinement of P_i for all i = 1, 2, 3, ...

Proposition 5.1.11 implies the following result.

Proposition 5.1.12 *A subsequence of a decreasing (converging and/or refined) sequence of tagged partitions is also a decreasing (converging and/or refined) sequence of tagged partitions.*

Lemma 5.1.13 implies the following result.

Proposition 5.1.13 *Any refined sequence of tagged partitions is decreasing.*

Let us consider a gauge sequence $\boldsymbol{\Delta} = \{\delta_n; n = 1, 2, 3, \ldots\}$.

Definition 5.1.19 A sequence $\mathbf{L} = \{P_i; i = 1, 2, 3, \ldots\}$ of tagged partitions P_i is called $\boldsymbol{\Delta}$-*fine* if each partitions P_i is δ_n-fine for all $n = 1, 2, 3, \ldots$

Theorem 5.1.1 implies the following result.

Theorem 5.1.2 *For any interval* $[a, b]$ *with* $a<b$ *and any gauge sequence* $\boldsymbol{\Delta}$, *there is a* $\boldsymbol{\Delta}$-*fine sequence* $\boldsymbol{L}$ *of tagged partition of* $[a, b]$.

Definition 5.1.20 A gauge sequence $\boldsymbol{\Delta} = \{\delta_n; n = 1, 2, \ldots\}$ is called *decreasing* if $\delta_{n+1} \leq \delta_n$ for all $n = 1, 2, 3, \ldots$

Proposition 5.1.14 *If* $\boldsymbol{\Delta}$ *is a decreasing gauge sequence, then any subsequence of a* $\boldsymbol{\Delta}$-*fine sequence of tagged partitions is also a* $\boldsymbol{\Delta}$-*fine sequence of tagged partitions.*

Proof is left as an exercise.

5.2 Hyperintegration over Finite Intervals

Let us consider a finite interval $[a, b]$, total function $f : [a, b] \to \boldsymbol{R}$ and a finite tagged interval partition $P = \{([a_i, b_i], c_i); i = 1, 2, 3, \ldots, n\}$ of the interval $[a, b]$.

Definition 5.2.1 The *P-sum SP*(*f*) of the function f called the *Riemann sum* corresponding to the partition $P = \{[a_i, b_i]; i = 1, 2, 3, \ldots, n\}$ is equal to

$$\sum_{i=1}^{n} f(c_i)\boldsymbol{\Delta}_i$$

where $\boldsymbol{\Delta}_i = |b_i - a_i|$.

Note that a Riemann sum depends on several parameters: the function $f(x)$, partition P, and choice of points c_i.

Let us consider a sequence $\mathbf{P} = \{P_i; i \in \omega\}$ of finite tagged interval partitions $P_i = \{([a_{ij}, b_{ij}], c_{ij}); j = 1, 2, 3, \ldots, n_i\}$ of the interval $[a, b]$.

Definition 5.2.2

(a) The *partial* (*Riemann*) *hyperintegral* of the function $f(x)$ over the tagged interval partition sequence $\mathbf{P}$ is equal to

$$\int_{\mathbf{P}} f(x)\, \mathrm{d}x = \mathrm{Hn}(SP_i(f))_{i \in \omega}$$

(b) The partial (Riemann) hyperintegral $\int_{\mathbf{P}} f(x)\, \mathrm{d}x$ is called the *partial* (*Riemann*) *integral* of $f(x)$ over $\mathbf{P}$ if its value is a real number.

The integral is called partial because it is defined only for one partition.

As $f(x)$ is a total function, the partial sum $SP_i(f)$ is also always defined. Consequently, we have the following simple but basic result.

Theorem 5.2.1 *For any total function* $f : [a, b] \to \boldsymbol{R}$ *and any sequence* $\mathbf{P} = \{P_i; i \in \omega\}$ *of finite tagged interval partitions of* [a, b], *the partial hyperintegral of f over the sequence* $\mathbf{P}$ always exists.

It is possible that the partial hyperintegrals of the same function f over two different tagged partition sequences are not equal.

Example 5.2.1 Let us consider the real function $f(x) = x \cdot \chi_Q(x)$, namely,

$$f(x) = \begin{cases} x & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$$

and take the sequence $\mathbf{P} = \{P_i = \{[a_{ij}, b_{ij}]; j = 1, 2, 3, \ldots, i\}; i \in \omega\}$ of finite interval partitions of the interval [0, 1], which divides this interval into i equal intervals. Then we can correspond to this sequence two tagged interval partitions of the interval $[0, 1]$: $\mathbf{R} = \{R_i = \{([a_{ij}, b_{ij}], c_{ij}); c_{ij} \in [a_{ij}, b_{ij}]$ and c_{ij} is a rational number, $j = 1, 2, 3, \ldots, i\}; i \in \omega\}$, and $\mathbf{Q} = \{Q_i = \{([a_{ij}, b_{ij}], d_{ij}); d_{ij} \in [a_{ij}, b_{ij}]$ and d_{ij} is an irrational number, $j = 1, 2, 3, \ldots, i\}; i \in \omega\}$

Applying Definition 5.2.1, we have

$$\int_{\mathbf{R}} f(x)\ \mathrm{d}x = \frac{1}{2}$$

while

$$\int_{\mathbf{Q}} f(x)\ \mathrm{d}x = 0$$

Remark 5.2.1 Partial integration allows one to integrate not only total functions but also partial functions. It is important only that the integrated function f is defined at the tags of all tagged partitions from the partition sequence **P** over which this function is integrated. In particular, it is possible to apply such integration to functions defined on discrete sets of numbers.

Let us study properties of partial hyperintegrals and integrals.

Lemma 5.2.1 *If* **R** *is a tagged interval partition sequence of an interval* $[a, b]$ *and* $\int_{\mathbf{R}} f(x)\ \mathrm{d}x$ *is a real number, then* $\int_{\mathbf{R}} f(x)\ \mathrm{d}x = \int_{\mathbf{Q}} f(x)\ \mathrm{d}x$ *for any subsequence* **Q** *of* **R**.

The following result demonstrates that this result is not true when $\int_{\mathbf{R}} f(x)\ \mathrm{d}x$ is equal to an arbitrary hypernumber.

Theorem 5.2.2

(a) *If* $\alpha = \int_{\mathbf{R}} f(x)\ \mathrm{d}x$ *and* α *is either an infinitely increasing or infinite positive or infinite expanding hypernumber, then for any hypernumber* $\beta \in \boldsymbol{R}_\omega$, *there is a subsequence* **Q** *of* **R** *such that* $\int_{\mathbf{Q}} f(x)\ \mathrm{d}x > \beta$.

(b) *If* $\alpha = \int_{\mathbf{R}} f(x)\ \mathrm{d}x$ *and* α *is either an infinitely decreasing or infinite negative or infinite expanding hypernumber, then for any hypernumber* $\eta \in \boldsymbol{R}_\omega$, *there is a subsequence* **P** *of* **R** *such that* $\int_{\mathbf{P}} f(x)\ \mathrm{d}x < \eta$.

Proof is similar to the proof of Theorem 4.2.1.

Corollary 5.2.1 *If* $\alpha = \int_{\mathbf{R}} f(x)\,\mathrm{d}x$ *and* Spec α *is not bounded from above (from below), then for any hypernumber* $\beta \in \boldsymbol{R}_\omega$, *there is a subsequence* $\mathbf{Q}$ *of* $\mathbf{R}$ *such that* $\int_{\mathbf{Q}} f(x)\,\mathrm{d}x > \beta$ (*correspondingly* $\int_{\mathbf{Q}} f(x)\,\mathrm{d}x < \beta$).

Indeed, when *Spec* α is not bounded from above, then α is either an infinitely increasing hypernumber or infinite positive hypernumber or infinite expanding hypernumber and the statement directly follows from Theorem 5.2.2a.

When *Spec* α is not bounded from below, then α is either an infinitely decreasing hypernumber or infinite negative hypernumber or infinite expanding hypernumber and the statement directly follows from Theorem 5.2.2b.

Corollary 5.2.2 *If* $\alpha = \int_{\mathbf{R}} f(x)\,\mathrm{d}x$ *and* α *is an infinite oscillating hypernumber, then for any hypernumber* $\beta \in \boldsymbol{R}_\omega$, *there is a subsequence* $\mathbf{Q}$ *of* $\mathbf{R}$ *such that* $\int_{\mathbf{Q}} f(x)\,dx > \beta$ *and a subsequence* $\mathbf{P}$ *of* $\mathbf{R}$ *such that* $\int_{\mathbf{P}} f(x)\,\mathrm{d}x < \beta$.

Indeed, an infinite oscillating hypernumber is an infinite expanding hypernumber.

In Theorem 5.2.2, all partial hyperintegrals are infinite. However, it is possible that a function has both finite and infinite partial hyperintegrals over the same interval.

Example 5.2.2 Let us consider the total real function

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and two sequences $\mathbf{R} = \{R_i; i \in \omega\}$ of finite tagged interval partitions of the interval $[-1, 1]$ and sequence $\mathbf{P} = \{P_i; i \in \omega\}$ of finite tagged interval partitions of the interval $[-1, 1]$ where

$$R_1 = \left\{\left(\left[-1, -\frac{1}{2}\right], -\frac{1}{2}\right), \left(\left[-\frac{1}{2}, 0\right], -\frac{1}{2}\right), \left(\left[0, \frac{1}{2}\right], \frac{1}{2}\right), \left(\left[\frac{1}{2}, 1\right], \frac{1}{2}\right)\right\}$$

$$R_2 = \left\{\left(\left[-1, -\frac{1}{2}\right], -\frac{1}{2}\right), \left(\left[-\frac{1}{2}, -\frac{1}{4}\right], -\frac{1}{4}\right), \left(\left[-\frac{1}{4}, 0\right], -\frac{1}{4}\right),\right.$$
$$\left.\left(\left[0, \frac{1}{4}\right], \frac{1}{4}\right), \left(\left[\frac{1}{4}, \frac{1}{2}\right], \frac{1}{2}\right), \left(\left[\frac{1}{2}, 1\right], \frac{1}{2}\right)\right\}$$

$$R_n = \left\{\left(\left[-1, -\frac{1}{2}\right], -\frac{1}{2}\right), \left(\left[-\frac{1}{2}, -\frac{1}{4}\right], -\frac{1}{4}\right), \ldots, ([-(1/2^{n-1}), -(1/2^n)],\right.$$
$$-(1/2^n)), ([-(1/2^n), 0], -(1/2^n)), ([0, (1/2^n)], (1/2^n)), ([(1/2^n),$$
$$\left.\times (1/2^{2n-1})], (1/2^n)), \ldots, \left(\left[\frac{1}{4}, \frac{1}{2}\right], \frac{1}{4}\right), \left(\left[\frac{1}{2}, 1\right], -\frac{1}{2}\right)\right\}$$

and

$$P_1 = \left\{\left(\left[-1, -\frac{1}{2}\right], -\frac{1}{2}\right), \left(\left[-\frac{1}{2}, 0\right], -\frac{1}{2}\right), \left(\left[0, \frac{1}{4}\right], \frac{1}{4}\right), \times \left(\left[\frac{1}{4}, \frac{1}{2}\right], \frac{1}{4}\right), \left(\left[\frac{1}{2}, 1\right], \frac{1}{2}\right)\right\}$$

$$P_2 = \left\{\left(\left[-1, -\frac{1}{2}\right], -\frac{1}{2}\right), \left(\left[-\frac{1}{2}, 0\right], -\frac{1}{2}\right), \left(\left[0, \frac{1}{8}\right], \frac{1}{8}\right), \times \left(\left[\frac{1}{8}, \frac{1}{4}\right], \frac{1}{8}\right), \left(\left[\frac{1}{4}, \frac{1}{2}\right], \frac{1}{4}\right), \left(\left[\frac{1}{2}, 1\right], \frac{1}{2}\right)\right\}$$

$$P_n = \left\{\left(\left[-1, -\frac{1}{2}\right], -\frac{1}{2}\right), \left(\left[-\frac{1}{2}, 0\right], -\frac{1}{2}\right), ([0, (1/2^{n+1})], (1/2^{n+1})), ([(1/2^{n+1}), \times (1/2^n)], (1/2^{n+1})), \ldots, \left(\left[\frac{1}{4}, \frac{1}{2}\right], \frac{1}{4}\right), \left(\left[\frac{1}{2}, 1\right], -\frac{1}{2}\right)\right\}$$

By Definition 5.1.2,

$$SR_1(f) = (-2)\left(\frac{1}{2}\right) + (-2)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{2}\right) = 0$$

$$SR_2(f) = (-2)\left(\frac{1}{2}\right) + (-4)\left(\frac{1}{4}\right) + (-4)\left(\frac{1}{4}\right) + (4)\left(\frac{1}{4}\right) + (4)\left(\frac{1}{4}\right) + (2)\left(\frac{1}{2}\right) = 0$$

$$SR_n(f) = (-2)\left(\frac{1}{2}\right) + (-4)\left(\frac{1}{4}\right) + \cdots + (-2^n)(1/2^n) + (-2^n)(1/2^n) + (2^n)(1/2^n) + (2^n)(1/2^n) + \cdots + (4)\left(\frac{1}{4}\right) + (2)\left(\frac{1}{2}\right) = 0$$

So, in this case, we have

$$\int_{\mathbf{R}} f(x)\, \mathrm{d}x = 0$$

At the same time, by Definition 5.1.2,

$$SP_1(f) = (-2)\left(\frac{1}{2}\right) + (-2)\left(\frac{1}{2}\right) + (4)\left(\frac{1}{4}\right) + (4)\left(\frac{1}{4}\right) + (2)\left(\frac{1}{2}\right)$$
$$= -1 + (-1) + 1 + 1 + 1 = 1$$

$$SP_2(f) = (-2)\left(\frac{1}{2}\right) + (-2)\left(\frac{1}{2}\right) + (8)\left(\frac{1}{8}\right) + (8)\left(\frac{1}{8}\right) + (4)\left(\frac{1}{4}\right) + (2)\left(\frac{1}{2}\right) = 2$$

$$SP_n(f) = (-2)\left(\frac{1}{2}\right) + (-2)\left(\frac{1}{2}\right) + (2^{n+1})(1/2^{n+1}) + (2^{n+1})(1/2^{n+1}) + \cdots$$
$$+ (4)\left(\frac{1}{4}\right) + (2)\left(\frac{1}{2}\right) = n$$

So, in this case, the partial hyperintegral is equal to an infinite hypernumber, i.e.,

$$\int_{\mathbf{P}} f(x)\,\mathrm{d}x = \mathrm{Hn}(SP_i(f))_{i\in\omega} = \mathrm{Hn}(i)_{i\in\omega}$$

However, many properties of hyperintegrals are similar to properties of conventional integrals.

Theorem 5.2.3 *For any sequence* $\mathbf{P} = \{P_i; i \in \omega\}$ *of finite tagged interval partitions of* $[a, b]$, *any total in* $[a, b]$ *real function f and any real number c, we have*

$$\int_{\mathbf{P}} cf(x)\,\mathrm{d}x = c\int_{\mathbf{P}} f(x)\,\mathrm{d}x$$

Proof As $SP(cf) = \sum_{i=1}^{n} cf(c_i)\boldsymbol{\Delta}_I = (\sum_{i=1}^{n} f(c_i)\boldsymbol{\Delta}_i) = c(SP(f))$ for any real number c, by properties of hypernumbers, we have

$$\int_{\mathbf{P}} cf(x)\,\mathrm{d}x = \mathrm{Hn}(SP_i(cf))_{i\in\omega} = \mathrm{Hn}(c(SP_i(cf)))_{i\in\omega} = c(\mathrm{Hn}(SP_i(cf))_{i\in\omega}$$
$$= c\int_{\mathbf{P}} f(x)\,\mathrm{d}x$$

Theorem is proved.

Theorem 5.2.4 *For any sequence* $\mathbf{P} = \{P_i; i \in \omega\}$ *of finite tagged interval partitions of* $[a, b]$ *and any two total in* $[a, b]$ *real functions f and g, we have*

$$\int_{\mathbf{P}} (f(x) + g(x))\,\mathrm{d}x = \int_{\mathbf{P}} f(x)\,\mathrm{d}x + \int_{\mathbf{P}} g(x)\,\mathrm{d}x$$

Proof As $SP(f + g) = \sum_{i=1}^{n} (f + g)(c_i)\Delta_I = \sum_{i=1}^{n} (f(c_i) + g(c_i))\Delta_i = \sum_{i=1}^{n} (f(c_i)\Delta_i + g(c_i)\Delta_i) = \sum_{i=1}^{n} (f(c_i)\Delta_i + \sum_{i=1}^{n} g(c_i)\Delta_i) = SP(f) + SP(g)$ for any total in $[a, b]$ real functions f and g, we have

$$\int_{\mathbf{P}} (f(x) + g(x))\,\mathrm{d}x = \mathrm{Hn}(SP_i(f+g))_{i\in\omega} = \mathrm{Hn}(SP_i(f) + SP_i(g))_{i\in\omega}$$

$$= \mathrm{Hn}(SP_i(f))_{i\in\omega} + \mathrm{Hn}(SP_i(g))_{i\in\omega} = \int_{\mathbf{P}} f(x)\,\mathrm{d}x + \int_{\mathbf{P}} g(x)\,\mathrm{d}x$$

Theorem is proved.

Corollary 5.2.3 *For any sequence* $\mathbf{P} = \{P_i; i \in \omega\}$ *of finite tagged interval partitions of* $[a, b]$*, any two total in* $[a, b]$ *real functions f and g, and any real numbers c and d, we have*

$$\int_{\mathbf{P}} (cf(x) + dg(x))\,\mathrm{d}x = c\int_{\mathbf{P}} f(x)\,\mathrm{d}x + d\int_{\mathbf{P}} g(x)\,\mathrm{d}x$$

Corollary 5.2.4 *For any sequence* $\mathbf{P} = \{P_i; i \in \omega\}$ *of finite tagged interval partitions of* $[a, b]$*, the partial Riemann hyperintegral* $\int_{\mathbf{P}}$ *is a linear hyperfunctional in the space of all real functions.*

Note that if $\int_{\mathbf{P}}$ is always equal to a real number, then it is a linear functional in the space of all real functions.

Theorem 5.2.5 *For any sequence* $\mathbf{P} = \{P_i; i \in \omega\}$ *of finite tagged interval partitions of* $[a, b]$ *and any two total in* $[a, b]$ *real functions f and g such that* $f \leq g$*, we have*

$$\int_{\mathbf{P}} f(x)\,\mathrm{d}x \leq \int_{\mathbf{P}} g(x)\,\mathrm{d}x$$

Proof As $f \leq g$ implies $SP_i(f) \leq SP_i(g)$ for all $i = 1, 2, 3, \ldots$, by properties of hypernumbers, we have

$$\int_{\mathbf{P}} f(x)\,\mathrm{d}x = \mathrm{Hn}(SP_i(f))_{i\in\omega} \leq \mathrm{Hn}(SP_i(g))_{i\in\omega} = \int_{\mathbf{P}} g(x)\,\mathrm{d}x$$

Theorem is proved.

Corollary 5.2.5 *For any sequence* $\mathbf{P} = \{P_i; i \in \omega\}$ *of finite tagged interval partitions of* $[a, b]$ *and any positive (negative) total in* $[a, b]$ *real function f,* $\int_{\mathbf{P}} f(x)\,dx \geq 0$ *(correspondingly* $\int_{\mathbf{P}} f(x)\,dx \leq 0$*).*

Let us take a sequence $\mathbf{P} = \{P_i; i \in \omega\}$ of finite tagged interval partitions $P_i = \{([a_{ij}, b_{ij}], c_{ij}); j = 1, 2, 3, \ldots, n_i\}$ of the interval $[a, b]$, a sequence $\mathbf{R} = \{R_i; i \in \omega\}$ of finite tagged interval partitions $R_i = \{([u_{ij}, v_{ij}], d_{ij}); j = 1, 2, 3, \ldots, m_i\}$ of the interval $[b, c]$ and the sequence $\mathbf{Q} = \{P_i \bigcup R_i; i \in \omega\}$. By Corollary 5.1.3, each $P_i \bigcup R_i (i \in \omega)$ is an interval partition of the interval $[a, c]$. Thus, $\mathbf{Q}$ is also a partition sequence and it is possible to integrate any real function f by this sequence.

Theorem 5.2.6 *For any sequences* $\mathbf{P} = \{P_i; i \in \omega\}$ *of finite tagged interval partitions of the interval* $[a, b]$ *and* $\boldsymbol{R} = \{R_i; i \in \omega\}$ *of finite tagged interval partitions of the interval* $[b, c]$ *and any total in* $[a, b]$ *real function f, we have*

$$\int_{\mathbf{Q}} f(x)\, \mathrm{d}x = \int_{\mathbf{P}} f(x)\, \mathrm{d}x + \int_{\mathbf{R}} f(x)\, \mathrm{d}x$$

Proof By properties of real hypernumbers, we have

$$\begin{aligned}\int_{\mathbf{Q}} f(x)\, \mathrm{d}x = \mathrm{Hn}(SQ_i(f))_{i\in\omega} = \mathrm{Hn}(SP_i(f) + SR_i(f))_{i\in\omega} &= \mathrm{Hn}(SP_i(f))_{i\in\omega} \\ &+ \mathrm{Hn}(SR_i(f))_{i\in\omega} \\ &= \int_{\mathbf{P}} f(x)\, \mathrm{d}x + \int_{\mathbf{R}} f(x)\, \mathrm{d}x\end{aligned}$$

Theorem is proved.

Theorems 5.2.3–5.2.6 describe properties of hyperintegrals that are similar to properties of conventional integrals. Now we demonstrate that hyperintegrals can have properties essentially different from properties of conventional integrals.

Theorem 5.2.7 *There is a real function f(x) defined in an interval* $[-1, 1]$ *such that for any hypernumber* β*, there is a sequence* $\mathbf{P} = \{P_i; i \in \omega\}$ *of finite tagged interval partitions of the interval* $[-1, 1]$ *such that*

$$\int_{\mathbf{P}} f(x)\, \mathrm{d}x = \beta$$

Proof We show that the function $f(x) = 1/x$ when $x \neq 0$ and $f(x) = 0$ when $x = 0$ (cf. Example 5.2.2) has the necessary properties. To do this, we prove the following lemma.

Lemma 5.2.2 *For the considered function f(x) and any real number a, there is a tagged interval partition* P_a *of the interval* $[-1, 1]$ *such that* $SP_a(f) = a$.

Proof At first, we notice that such a partition P_a exists when a is natural number n. Indeed, the partition P_n from Example 5.2.2 satisfies this condition.

To build the necessary partition P_a in a general case when the number a is positive, we slightly change the partition P_n from Example 5.2.2. Namely, we define

$$\begin{aligned}P_a = \Big\{&\left(\left[-1, -\frac{1}{2}\right], -\frac{1}{2}\right), \left(\left[-\frac{1}{2}, 0\right], -\frac{1}{2}\right), ([0, (1/2^{n+1})], c), ([(1/2^{n+1}), \\ &(1/2^{2n})], (1/2^{n+1})), \ldots, \left(\left[\frac{1}{4}, \frac{1}{2}\right], \frac{1}{4}\right) \left(\left[\frac{1}{2}, 1\right], \frac{1}{2}\right)\Big\}\end{aligned}$$

where the point c is chosen so that it satisfies the following equation

$$(1/x)(1/2^{2n+2}) = 1 + u$$

where $u = a - n$ and $n = [a]$ is equal to the largest integer number that is less than a.

From this equation, we have

$$c = (1/1 + u)(1/2^{2n+2})$$

As $1 + u > 1$, the number c satisfies the inequality $c < 1/2^{2n+2}$. It means that in the interval $[0, 1/2^{2n+2}]$, there is a point c such that $(1/c)(1/2^{2n+2}) = 1 + u$. Thus,

$$\begin{aligned} SP_a(f) &= (-2)\left(\frac{1}{2}\right) + (-2)\left(\frac{1}{2}\right) + (2^{n+1})(f(c)) + (2^{n+1})(1/2^{n+1}) + \cdots \\ &\quad + (4)\left(\frac{1}{4}\right) + (2)\left(\frac{1}{2}\right) \\ &= n - 1 + (1 + u) = n + u = a \end{aligned}$$

When the number a is negative, i.e., $a = -d$ and $d > 0$, we take the partition dual to the partition P_d. Namely, we define

$$\begin{aligned} P_a = \Bigg\{ &\left(\left[-1, -\frac{1}{2}\right], -\frac{1}{2}\right), \left(\left[-\frac{1}{2}, -\frac{1}{4}\right], -\frac{1}{4}\right), \ldots, ([-(1/2^n), -(1/2^{2n+1})], \\ &-(1/2^{n+1})), ([-(1/2^{n+1}), 0], c), \left(\left[0, \frac{1}{2}\right], \frac{1}{2}\right), \left(\left[\frac{1}{2}, 1\right], \frac{1}{2}\right) \Bigg\} \end{aligned}$$

Similar considerations as before show that

$$SP_a(f) = a$$

Lemma is proved.

Now using this lemma and taking a hypernumber $\alpha = \mathrm{Hn}(a_i)_{i \in \omega}$, we build a sequence $\mathbf{P}$ of tagged partitions Q_i of the interval $[-1, 1]$ such that

$$\begin{aligned} SQ_i(f)) = a_i \quad &\text{for all } i = 1, 2, 3, \ldots \\ &SQ(f) = SP(f_d) \end{aligned}$$

As a result, we obtain

$$\int_{\mathbf{P}} f(x)\, \mathrm{d}x = \mathrm{Hn}(SQ_i(f))_{i \in \omega} = \mathrm{Hn}(a_i)_{i \in \omega} = \alpha$$

Theorem is proved.

Moreover, the result of Theorem 5.2.7 is also valid for the following infinite family of real functions $\boldsymbol{F} = \{f_a(x); a \in \boldsymbol{R}\}$.

Corollary 5.2.6 *For any real number a, the function $f_a(x)$ is equal to a/x when $x \neq 0$ and equal to 0 when $x = 0$ and any real hypernumber α, there is a sequence* **P** *of tagged partitions of the interval* $[-1, 1]$ *such that* $\int_{\mathbf{P}} f(x)\, dx = \alpha$.

Indeed, by Theorem 5.2.7, there is a sequence **P** of tagged partitions of the interval $[-1, 1]$ such that $\int_{\mathbf{P}} f(x)\, \mathrm{d}x = \alpha/a$. We see that $f_a(x) = af(x)$ for the function $f_a(x)$ determined in the proof of Theorem 5.2.7. Then by Theorem 5.2.3, we have

$$\int_{\mathbf{P}} f_a(x)\, \mathrm{d}x = a\left(\int_{\mathbf{P}} f(x)\, \mathrm{d}x\right) = \alpha\Big)$$

It is possible to obtain the same result for any interval $[-d, d]$ where d is an arbitrary positive real number.

Corollary 5.2.7 *There is a function $f(x)$ such that for any real number α, there is a sequence* **P** *of tagged partitions of the interval* $[-d, d]$ *such that* $\int_{\mathbf{P}} f(x)\, \mathrm{d}x = \alpha$.

Proof Taking a number $\alpha = \mathrm{Hn}(a_i)_{i\in\omega}$, we build a sequence **P** of tagged partitions Q_i of the interval $[-d, d]$ such that

$$SQ_i(f)) = a_i \quad \text{for all } i = 1, 2, 3, \ldots$$

To do this, we define $f_d(x) = 1/(dx)$. Besides, we correspond the tagged partition Q of the interval $[-d, d]$ to a tagged partition $P = \{([a_i, b_i], q_i);\ i = 1, 2, 3, \ldots\}$ of the interval $[-1, 1]$ by the following formula:

$$Q = dP = \{([c_i, d_i], r_i); c_i = da_i, d_i = db_i, r_i = dq_i, ([a_i, b_i], q_i) \in P, i = 1, 2, 3, \ldots\}$$

By construction, as $d > 0$, we have

$$f(r_i)(|d_i - c_i|) = f(dq_i)(|db_i - da_i|) = f(q_i)(|b_i - a_i|)$$

Consequently, the tagged partition Q has the following property (C)

$$SQ(f) = SP(f_d)$$

Using this procedure, we build a sequence **P** of tagged partitions Q_i of the interval $[-d, d]$ such that $Q_i = dP_{a_i}$ where partitions P_{a_i} are defined in the proof of Theorem 5.2.7. As all partitions Q_i have property (C), we obtain

$$\int_{\mathbf{P}} f(x)\,\mathrm{d}x = \mathrm{Hn}(SQ_i(f))_{i\in\omega} = \mathrm{Hn}(a_i)_{i\in\omega} = \alpha$$

Corollary is proved.

Remark 5.2.2 It is possible to obtain the same result using only converging partitions because the considered function $f(x)$ is an odd function.

Evidently this result is not valid for an arbitrary real function. For instance, all partial hyperintegrals of a constant function are equal to a fixed real number or all partial hyperintegrals of a positive function are equal to a non-negative hypernumber. Thus, we have the following problems.

Problem 5.2.1 *Find conditions for a function f(x) such that the partial hyperintegral* $\int_{\mathbf{P}} f(x)\,\mathrm{d}x$ *over a partition sequence* **P** *for a given interval* $[a, b]$ *can be equal to any real hypernumber.*

Let us consider a sequence $\mathbf{P} = \{P_i; i \in \omega\}$ of finite tagged interval partitions $P_i = \{([a_{ij}, b_{ij}], c_{ij}); j = 1, 2, 3, \ldots, n_i\}$of the interval $[a, b]$.

Theorem 5.2.8 *The following conditions are equivalent:*

(a) *The partial hyperintegral* $\int_{\mathbf{P}} f(x)\,\mathrm{d}x$ *is a real number.*
(b) *Spec* $\int_{\mathbf{P}} f(x)\,\mathrm{d}x$ *consists of one point and Spec* $\int_{\mathbf{P}} f(x)\,\mathrm{d}x = \mathrm{E}Spec \int_{\mathbf{P}} f(x)\,\mathrm{d}x$.
(c) *The partial hyperintegral* $\int_{\mathbf{Q}} f(x)\,\mathrm{d}x$ *is a real number for any subsequence* **Q** *of the sequence* **P**.
(d) *All partial hyperintegrals* $\int_{\mathbf{Q}} f(x)\,\mathrm{d}x$ *coincide for all subsequences* **Q** *of the sequence* **P**.

Proof (a) ⇒ (b) because for any real number a, we have (by Proposition 2.1.8) $Spec\ a = \{a\}$ and $\mathrm{E}Spec\ a = \{a\}$.

(b) ⇒ (c) Let us assume that condition (b) is true and the hyperintegral $\int_{\mathbf{Q}} f(x)\,\mathrm{d}x = \alpha \notin \boldsymbol{R}$ for some subsequence **Q** of the sequence **P**, while *Spec* $\int_{\mathbf{P}} f(x)\,\mathrm{d}x$ consists of one point and *Spec* $\int_{\mathbf{P}} f(x)\,\mathrm{d}x = ESpec \int_{\mathbf{P}} f(x)\,\mathrm{d}x$. Then by Theorem 2.1.2, α is either bounded oscillating hypernumber or infinite hypernumber. In the first case, by Proposition 2.1.7, *Spec* α contains, at least, two points. As the hypernumber α is a subhypernumber of the hypernumber $\int_{\mathbf{P}} f(x)\,\mathrm{d}x$, by Proposition 2.2.12, $Spec\,\alpha \subseteq Spec \int_{\mathbf{P}} f(x)\,\mathrm{d}x$. Thus, *Spec* $\int_{\mathbf{P}} f(x)\,\mathrm{d}x$ contains more than one point. This contradicts our assumption that condition (b) is true and shows that α cannot be a bounded oscillating hypernumber.

When α is an infinite hypernumber, we have two options: either α is an infinite monotone hypernumber or infinite oscillating hypernumber. In the first case, E*Spec* α consists of one point (either ∞ or −∞, cf. Chap. 2), while *Spec* α is empty. As $Spec\,\alpha \subseteq Spec \int_{\mathbf{P}} f(x)\,\mathrm{d}x$ and $Spec \int_{\mathbf{P}} f(x)\,\mathrm{d}x \subseteq \mathrm{E}Spec \int_{\mathbf{P}} f(x)\,\mathrm{d}x$, we see that both cases $\mathrm{E}Spec \int_{\mathbf{P}} f(x)\,\mathrm{d}x$ contains more points than $Spec \int_{\mathbf{P}} f(x)\,\mathrm{d}x$. This also contradicts condition (b).

When α is an infinite oscillating hypernumber. The same reasoning gives us that $\mathrm{E}Spec \int_{\mathbf{P}} f(x)\,\mathrm{d}x$ contains more points than $Spec \int_{\mathbf{P}} f(x)\,\mathrm{d}x$. This contradicts condition (b).

Thus, when condition (c) is not valid, condition (b) is also invalid and by the Law of Contraposition for propositions (cf., for example, Church 1956), (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (d). Let us take a subsequence $\mathbf{Q} = \{P_j; j \in \omega\}$ of the sequence $\mathbf{P}$. As $\int_{\mathbf{P}} f(x)\,\mathrm{d}x$ is a real number a, by Proposition 2.1.3, $\lim_{i\to\infty} SP_i(f) = a$. In the same way, if $\int_{\mathbf{Q}} f(x)\,\mathrm{d}x$ is a real number b, then $\lim_{i\to\infty} SP_j(f) = b$. Consequently, $a = b$ as $\{SP_j(f); j = 1, 2, 3, \ldots\}$ is a subsequence of the sequence $\{SP_i(f); i = 1, 2, 3, \ldots\}$ and a convergent sequence and any of its subsequence have the same limit. As $\mathbf{Q}$ is an arbitrary subsequence of $\mathbf{P}$, we obtain the necessary implication (c) $\Rightarrow$ (d).

(d) $\Rightarrow$ (a). If condition (a) is not valid, then the hyperintegral $\int_{\mathbf{P}} f(x)\,\mathrm{d}x = \alpha \notin \boldsymbol{R}$, and by Propositions 2.2 and 2.3, α is either an infinite hypernumber or a finite oscillating hypernumber. In the latter case, by Theorem 5.2.2, there is a subsequence $\mathbf{Q}$ of the sequence $\mathbf{P}$ such that $\int_{\mathbf{P}} f(x)\,\mathrm{d}x \neq \int_{\mathbf{Q}} f(x)\,\mathrm{d}x$.

In the first case, by Proposition 2.1.7, *Spec* α contains, at least, two points, say, c and d. Then there is a subsequence $\mathbf{Q}$ of the sequence $\mathbf{P}$ such that $\int_{\mathbf{Q}} f(x)\,\mathrm{d}x = c$. Consequently, $\int_{\mathbf{P}} f(x)\,\mathrm{d}x \neq \int_{\mathbf{P}} f(x)\,\mathrm{d}x$. It means that condition (d) is also invalid. Thus, by the Law of Contraposition (cf. Church 1956), we have (d) $\Rightarrow$ (a).

So, (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

Theorem is proved.

Corollary 5.2.8 *If partial hyperintegral* $\int_{\mathbf{P}} f(x)\,\mathrm{d}x$ *is not a real number, then for some subsequence* $\mathbf{Q}$ *of the sequence* $\mathbf{P}$.

$$\int_{\mathbf{Q}} f(x)\,\mathrm{d}x \neq \int_{\mathbf{P}} f(x)\,\mathrm{d}x$$

One of the basic results in the calculus is the Newton-Leibniz theorem. Here we prove a counterpart of this theorem in much broader context, making the classical Newton-Leibniz theorem a particular case of the new extended Newton-Leibniz theorem.

Let us consider a real function $F : [a, b] \to \boldsymbol{R}$ and $\mathbf{P} = \{P_n; n \in \omega\}$ a converging refined sequence of right-tagged interval partitions $P_n = \{([c_{n,i}, d_{n,i}], d_{n,i}); i = 1, 2, 3, \ldots, m_n\}_{n,i}); i = 1, 2, 3, \ldots, m_n\}$ of the interval [a, b]. For a point c from the interval [a, b], it possible to consider the set $R_c = \{(\langle c_{n,i}, d_{n,i}\rangle; c \in [c_{n,i}, d_{n,i}] \in P_n, c_{n,i} < c \leq d_i, n = 1, 2, 3, \ldots\}$. As the sequence $\mathbf{P}$ is converging and refined, R_c is an A-approximation of the point c and it is possible to take the sequential partial derivative $\partial/_{\partial R_c} F(c)$. We call it the *right-tagged sequential partial derivative* of $F(x)$ at the point c defined by the sequence $\mathbf{P}$. Taking such a sequential partial derivative at each point x of the interval [a, b], we obtain a restricted extrafunction $f(x) = \partial/_{\partial R_x} F(x)$.

In a similar way, if $\mathbf{Q} = \{Q_n; n \in \omega\}$ a converging refined sequence of left-tagged interval partitions of the interval [a, b], then it is possible to define the *left-tagged sequential partial derivative* of $F(x)$ defined by the sequence $\mathbf{Q}$.

By definition, $f(x) = \partial/_{\partial R_x}F(x) = \mathrm{Hn}((F(d_{n,i}) - F(c_{n,i}))/(d_{n,i} - c_{n,i}))_{n\in\omega}$ $= Hn(\Delta_{n,i}{}^{\mathbf{P}}F/\Delta_{n,i}{}^{\mathbf{P}}x)_{n\in\omega}$ where $\Delta_{n,i}{}^{\mathbf{P}}x = d_{n,i} - c_{n,i}$ for the pair $\langle c_{n,i}, d_{n,i}\rangle$ from R_x such that $x \in [c_{n,i}, d_{n,i}] \in P_n, c_{n,i}<x$ and $\Delta_{n,i}{}^{\mathbf{P}}F = F(d_{n,i}) - F(c_{n,i})$. Thus, we can define the nth Riemann sum $SP_n(f) = \sum_{i=1}^{m_n} f_n(d_{n,i})\Delta_{n,i}$ where $f_n(d_{n,i}) = (F(d_{n,i}) - F(c_{n,i}))/(d_{n,i} - c_{n,i})$ and m_n is the number of intervals in the partition P_n. Riemann sums allow us to construct the partial Riemann hyperintegral

$$\int_{\mathbf{P}} f(x)\,\mathrm{d}x = \mathrm{Hn}(SP_n(f))_{n\in\omega}$$

Note that this is the hyperintegral of the extrafunction.

Theorem 5.2.9 (Integral Newton-Leibniz Theorem) *If $F : [a, b] \to \boldsymbol{R}$ is a real function, $\mathbf{P} = \{P_i; i \in \omega\}$ is a converging refined sequence of right-tagged (left-tagged) interval partitions of the interval $[a, b]$ and at all points x from $[a, b]$, f(x) is a right-tagged (left-tagged) sequential partial derivative of F(x) defined by the sequence $\mathbf{P}$, then*

$$\int_{\mathbf{P}} f(x)\,\mathrm{d}x = F(b) - F(a)$$

Proof At first, let us simplify the nth Riemann sum $SP_n(f)$. By definition, $P_n = \{([c_{n,i}, d_{n,i}], d_{n,i}); i = 1, 2, \ldots, m_n\}$. Consequently, we have

$$\begin{aligned} SP_n(f) &= \sum_{i=1}^{m_n} f_n(d_{n,i})\Delta_{n,i} = \sum_{i=1}^{m_n} ((F(d_{n,i}) - F(c_{n,i}))/\Delta_{n,i})\Delta_{n,i} \\ &= \sum_{i=1}^{m_n} (F(d_{n,i}) - F(c_{n,i})) = F(d_{m,n}) - F(c_{n,1}) = F(b) - F(a) \end{aligned}$$

because $c_{n,1} = a, d_{n,1} = c_{n,2}, d_{n,2} = c_{n,3}, \ldots, d_{n,m_{n-1}} = c_{n,m_n}$, and $d_{n,m_n} = b$.

Thus, by Theorem 5.2.4, we have

$$\int_{\mathbf{P}} f(x)\,\mathrm{d}x = \mathrm{Hn}(SP_n(f))_{n\in\omega} = \mathrm{Hn}(F(b) - F(a))_{n\in\omega} = F(b) - F(a)$$

For a converging sequence $\mathbf{Q} = \{Q_i; i \in \omega\}$ of left-tagged interval partitions of the interval $[a, b]$ and a left-tagged sequential partial derivative $g(x)$ of $F(x)$, the proof is similar.

Theorem is proved.

Given a real function $F : [a, b] \to \boldsymbol{R}$ and a converging sequence $\mathbf{P} = \{P_i; i \in \omega\}$ of right-tagged interval partitions of the interval $[a, b]$, a point x from $[a, b]$, it is possible to build a right-tagged sequential partial derivative $f(x)$ of $F(x)$ defined by the sequence $\mathbf{P}$ (as it is done above) and to form the hyperintegral of f over the interval [a, x] with respect to the sequence **P**. Namely, taking the partition $P_n = \{([c_{n,i}, d_{n,i}], d_{n,i}); i = 1, 2, 3, \ldots, m_n\}$ from **P**, we put $SP_{n,x}(f) = \sum_{d_i \leq x} f(d_i)\Delta_i$ and define

$$\int_{\mathbf{P},x} f(x)\ \mathrm{d}x = \mathrm{Hn}(SP_{n,x}(f))_{n\in\omega}$$

Lemma 5.2.3 *For any real function* $F : [a, b] \to \boldsymbol{R}$ *and any converging refined sequence* $\mathbf{P} = \{P_i; i \in \omega\}$ *of right-tagged interval partitions of the interval* [a, b], *we have*

$$\int_{\mathbf{P},b} f(x)\ \mathrm{d}x = \int_{\mathbf{P}} f(x)\ \mathrm{d}x$$

Proof is left as an exercise.

At the same time, if $z \in [a, b]$ and $\mathbf{P} = \{P_n; n \in \omega\}$ is a converging sequence of right-tagged interval partitions of the interval [a, b], then the set $R = \{\langle c_{n,i}, d_{n,i}\rangle; z \in [c_{n,i}, d_{n,i}] \in P_n, c_{n,i} < z, n \in \omega\}$ is an A-approximation of z because **P** is a converging sequence. Thus, given a restricted extrafunction $G(x) = \mathrm{Hn}(g_n(x))_{n\in\omega}$, it is possible to define its sequential partial derivative at z by the following formula

$$\partial/_{\partial R}G(z) = \mathrm{Hn}((g_n(d_{n,i}) - g_n(c_{n,i}))/(d_{n,i} - c_{n,i}))_{n\in\omega}$$

In particular, when $z = b$, we have

$$\partial/_{\partial R}G(z) = \mathrm{Hn}((g_n(z) - g_n(c_{n,m_n}))/(z - c_{n,m_n}))_{n\in\omega}$$

Theorem 5.2.10 (Differential Newton-Leibniz Theorem)

(a) *If* $f : [a, b] \to \boldsymbol{R}$ *is a real function and* $\mathbf{P} = \{P_n; n \in \omega\}$ *is a converging refined sequence of right-tagged interval partitions of the interval* [a, z], *then*

$$\partial/_{\partial R}\int_{\mathbf{P}} f(x)\ \mathrm{d}x = f(z)$$

(b) *If a real function* $f : [a, b] \to \boldsymbol{R}$ *is continuous from the left at a point* z *from* [a, b] *and* $\mathbf{Q} = \{Q_n; n \in \omega\}$ *is a converging refined sequence of left-tagged interval partitions of the interval* [a, z], *then*

$$\partial/_{\partial R}\int_{\mathbf{Q}} f(x)\ \mathrm{d}x = f(z)$$

Proof (a) Let us denote $\int_{\mathbf{P}} f(x)\,\mathrm{d}x = \mathrm{Hn}(SP_n(f))_{n\in\omega}$ by $G(z) = \mathrm{Hn}(g_n(z))_{n\in\omega}$ where $g_n(z) = SP_n(f)$. Then

$$\begin{aligned} g_n(z) - g_n(d_{n,m_{n-1}}) &= g_n(z) - g_n(c_{n,m_n}) \\ &= \sum_{i-1}^{m_n} f(d_{n,i})\Delta_{n,i} - \sum_{i=1}^{m_n-1} f(d_{n,i})\Delta_{n,i} = f(d_{n,m_{n-1}})\Delta_{n,m_n} \\ &= f(z)\Delta_{n,m_n} \end{aligned}$$

Consequently,

$$\begin{aligned} (g_n(z) - g_n(c_{n,m_n}))/(z - c_{n,m_n}) &= (g_n(z) - g_n(d_{n,m_{n-1}}))/(d_{n,m_n} - c_{n,m_n}) \\ &= (f(z)\Delta_{n,m_n})/(d_{n,m_n} - c_{n,m_n}) = f(z) \end{aligned}$$

because by construction, $z = d_{n,m_n}, d_{n,m_n-1} = c_{n,m_n}$ and $\Delta_{n,m_n} = (d_{n,m_n} - c_{n,m_n})$ for all $n = 1, 2, 3, \ldots$. Thus,

$$\partial/_{\partial R} \int_{\mathbf{P}} f(x)\,\mathrm{d}x = \mathrm{Hn}((g_n(z) - g_n(c_{n,m_n}))/(z - c_{n,m_n}))_{n\in\omega} = \mathrm{Hn}(f(z))_{n\in\omega} = f(z)$$

(b) In a similar way, for the converging refined sequence $\mathbf{Q} = \{Q_n; n \in \omega\}$ of left-tagged interval partitions of the interval $[a, z]$, we have

$$g_n(z) - g_n(c_{n,m_n}) = \sum_{i=1}^{m_n} f(c_{n,i})\Delta_{n,i} - \sum_{i=1}^{m_n-1} f(c_{n,i})\Delta_{n,i} = f(c_{n,m_n})\Delta_{n,m_n}$$

Consequently,

$$\begin{aligned} (g_n(z) - g_n(c_{n,m_n}))/(z - -c_{n,m_n}) &= g_n(z) - g_n(d_{n,m_n-1}))/ \\ &= (f(c_{n,m_n})\Delta_{n,m_n})/(z - c_{m_n}) = f(c_{m_n}) \end{aligned}$$

and

$$\partial/_{\partial R} \int_{\mathbf{Q}} f(x)\,\mathrm{d}x = \mathrm{Hn}((g_n(z) - g_n(c_{m_n}))/(z - c_{m_n}))_{n\in\omega} = \mathrm{Hn}(f(c_{m_n}))_{n\in\omega} = f(z)$$

because by construction, elements c_{m_n} converge to z and function f is continuous from the left at the point z.

Theorem is proved.

Theorems 5.2.9 and 5.2.10 imply the conventional Newton-Leibniz Theorem.

Let us consider a class ***K*** of sequences of tagged interval partitions of the interval $[a, b]$.

Definition 5.2.3

(a) A function *f*(x) is (*Riemann*) *hyperintegrable* in ***K*** over [a, b] if for any two sequences **P** and **Q** from ***K***, the hyperintegrals over **P** and over **Q** coincide.

(b) In this case, $\int_{\mathbf{P}} f(x)\, \mathrm{d}x$ is called the (*Riemann*) *hyperintegral* of *f*(x) over ***K***.

Theorem 5.2.7 implies the following result

Proposition 5.2.1 *If a class* ***K*** *of sequences of tagged interval partitions of the interval* [a, b] *with each sequence contains its subsequences, then a function f(x) is (Riemann) hyperintegrable in* ***K*** *over* [a, b] *only if for any sequence* **P** *from* ***K***, *the hyperintegral* $\int_{\mathbf{P}} f(x)\, \mathrm{d}x$ *is a real number.*

Proof is left as an exercise.

Let us consider how this new concept of integration is related to the well-known constructions of integrals. The most familiar is the *Riemann integral* or *definite Riemann integral* of the function *f*(x) over the interval [a, b].

Let us consider the set ***CS*** of converging sequences $\mathbf{L} = \{P_i; i = 1, 2, 3, \ldots\}$ of tagged partitions of an interval [a, b].

Definition 5.2.4 (any course of the calculus Ross 1996)

(a) When for any sequences **L** and **H** from ***CS*** the limits $\lim_{i\to\infty}\{SP_i; P_i \in \mathbf{H}\}$ and $\lim_{i\to\infty}\{SR_i; R_i \in \mathbf{L}\}$ exist and are equal, then such a limit I is called the *Riemann integral* or *definite Riemann integral* of the function *f*(x) over the interval [a, b] and is denoted by

$$\int_a^b f(x)\, \mathrm{d}x \quad \text{or} \quad \int_a^b f\, \mathrm{d}x$$

In this case, the function *f*(x) is called the *integrand*, the numbers a and b are called the *bounds of integration*, and the interval [a, b] is called the *interval of integration*.

(b) If such an I exists, we say that f is *Riemann integrable* on [a, b].

By the definition of a limit, we have the following result.

Lemma 5.2.1 (any course of calculus) *The Riemann integral of the function f*(x) *over the interval* [a, b] *is a unique number when it exists.*

Properties of hyperintegrals show that the Riemann integral is a particular case of the Riemann hyperintegral.

Theorem 5.2.11 *A function f is Riemann integrable on the interval* [a, b] *if and only if* $\int_{\mathbf{P}} f(x)\, \mathrm{d}x$ *is one and the same real number for all converging sequences* ***P*** *of tagged interval partitions of* [a, b], *i.e., the function f(x) is (Riemann) hyperintegrable in* ***CS*** over [a, b].

Proof

Sufficiency. Let us assume that there is a real number d such that for any converging sequence **P** of tagged interval partitions of $[a, b]$, $\int_{\mathbf{P}} f(x)\, \mathrm{d}x = d$. Then by Proposition 2.1.3, $\lim_{i\to\infty} SP_i(f) = a$ for any converging sequence **P** of tagged interval partitions P_i of $[a, b]$. Thus, by Definition 5.2.4, the function f is Riemann integrable on the interval $[a, b]$ and

$$\int_a^b f(x)\ \mathrm{d}x = \int_{\mathbf{P}} f(x)\ \mathrm{d}x$$

Necessity. If a function f is Riemann integrable on the interval $[a, b]$, then all limits $\lim_{i\to\infty}\{\mathrm{S}P_i; P_i \in \mathbf{P}\}$ and $\lim_{i\to\infty}\{\mathrm{S}R_i; R_i \in \mathbf{Q}\}$ exist and are equal for any two converging sequences $\mathbf{P}$ and $\mathbf{Q}$ of tagged interval partitions of $[a, b]$. By Definition 5.2.1, we have

$$\int_{\mathbf{P}} f(x)\ \mathrm{d}x = \int_{\mathbf{Q}} f(x)\ \mathrm{d}x$$

Theorem is proved.

This result allows us to obtain many (known) properties of the Riemann integral as particular cases of properties of partial hyperintegrals.

Corollary 5.2.9 *For any total in* $[a, b]$ *real function f and any real number c, we have*

$$\int_a^b cf(x)\ \mathrm{d}x = c\int_a^b f(x)\ \mathrm{d}x$$

Corollary 5.2.10 *For any two total in* $[a, b]$ *real functions f and g, we have*

$$\int_a^b (f(x) + g(x))\ \mathrm{d}x = \int_a^b f(x)\ \mathrm{d}x + \int_a^b g(x)\ \mathrm{d}x$$

Corollary 5.2.11 *For any two total in* $[a, b]$ *real functions f and g and any real numbers c and d, we have*

$$\int_a^b (cf(x) + dg(x))\ \mathrm{d}x = c\int_a^b f(x)\ \mathrm{d}x + d\int_a^b g(x)\ \mathrm{d}x$$

Corollary 5.2.12 *For the Riemann integral is a linear functional in the space of all real functions.*

Corollary 5.2.13 *For any two total in* $[a, b]$ *real functions f and g such that* $f \leq g$*, we have*

$$\int_a^b f(x)\ \mathrm{d}x \leq \int_a^b g(x)\ \mathrm{d}x$$

Corollary 5.2.14 *For any positive (negative) total in* $[a, b]$ *real function f,* $\int_a^b f(x)\, dx \geq 0 (\int_a^b f(x)\, dx \leq 0)$.

Now let us explore relations of partial hyperintegrals with the gauge or Henstock–Kurzweil integral because it gives the most general schema of integration (Henstock 1988; Kurzweil 2002).

Let us consider a finite interval $[a, b]$ and total function $f : [a, b] \to \boldsymbol{R}$.

Definition 5.2.5

(a) A real number I is the *Henstock–Kurzweil* (*gauge*) *integral* of f if for every $\varepsilon > 0$, there exists a gauge δ such that whenever a partition P of $[a, b]$ is δ-fine, we have

$$|SP(f) - I| < \varepsilon$$

(b) If such a number I exists, f is called *Henstock–Kurzweil* (*gauge*) *integrable* on $[a, b]$.

Taking a total real function $f : [a, b] \to \boldsymbol{R}$, let us consider the following condition G.

There is a gauge sequence $\boldsymbol{\Delta} = \{\delta_n; n = 1, 2, 3, \ldots\}$ and $\boldsymbol{\Delta}$-fine tagged interval partition sequence $\mathbf{P} = \{P_i; i = 1, 2, 3, \ldots\}$ such that $\int_{\mathbf{P}} f(x)\, \mathrm{d}x = a$ is a real number and for any δ_n-fine partition R, we have

$$|SR(f) - a| < 1/n$$

Theorem 5.2.12 *A function f is Henstock–Kurzweil integrable on* $[a, b]$ *if and only if f satisfies Condition G.*

Proof

Necessity. Let us take a Henstock–Kurzweil integrable on $[a, b]$ real function f. Then its Henstock–Kurzweil integral I over $[a, b]$ exists and for each $n = 1, 2, 3, \ldots$, there is a gauge δ_n such that whenever R is δ_n-fine, we have

$$|SR(f) - I| < 1/n$$

In such a way, we obtain the gauge sequence $\boldsymbol{\Delta} = \{\delta_n; n = 1, 2, 3, \ldots\}$ from condition G. Taking any $\boldsymbol{\Delta}$-fine tagged interval partition sequence $\mathbf{P} = \{P_i; i = 1, 2, 3, \ldots\}$, we see that $\int_{\mathbf{P}} f(x)\, \mathrm{d}x = I$. So, condition G is satisfied for f.

Sufficiency. Let us assume that a real function f satisfies condition G. Then taking the number a from this condition, we see that it is the Henstock–Kurzweil integral I of f over $[a, b]$. Indeed, given $\varepsilon > 0$, we can find a natural number n such that $1/n < \varepsilon$. Then whenever a partition P of $[a, b]$ is δ_n-fine, we have

$$|SP_i(f) - I| < 1/n < \varepsilon$$

Theorem is proved.

As in the case of the Riemann integral, this result allows one to deduce many properties of the gauge integral from properties of partial hyperintegrals.

In addition, this result means that the Lesbegue integral is a special case of partial Riemann hyperintegrals because the gauge integral encompasses the Lesbegue integral.

Here we consider only extended integration (hyperintegration) of real functions. Integration and hyperintegration of extrafunctions is studied elsewhere.

5.3 Hyperintegration over Infinite Intervals

There are two approaches to hyperintegration in infinite spaces: extension and generation. In extension, a cumulative cover or B-approximation of an infinite space is taken so that elements of this cover/approximation are finite spaces (in our case, finite intervals), in which some integration schema, e.g., Riemann or Lesbesgue integral, works. Then this integration schema is extended to hyperintegration in the whole space. This approach is developed by Burgin (1990, 1995, 2004, 2008/2009).

In the generation approach, integrals are constructed by the technique used for hyperintegration in finite spaces, which is described in the previous section. Here we consider mostly the generation technique applied to hyperintegration of real functions. As a result, we have to consider only five types of infinite spaces, which means, in this context, spaces (intervals) that have the infinite measure. Namely, we have intervals $(-\infty, \infty) = \boldsymbol{R}, (-\infty, a), (a, \infty), (-\infty, a]$, and $[a, \infty)$. Besides, we explore only hyperintegration over $\boldsymbol{R}$ because it is possible to treat other cases in a similar way.

Let us consider a cumulative interval cover $R = \{[a_i, b_i]; i = 1, 2, 3, \ldots\}$ of $\boldsymbol{R}$ and a sequence $\mathbf{P} = \{P_i; i \in \omega\}$ where $P_i = \{([a_{ij}, b_{ij}], c_{ij}); j = 1, 2, 3, \ldots, n_i\}$ is a finite tagged interval partitions of the interval $[a_i, b_i]$. In this case, we have $a_{i+1} \leq a_i \leq b_i \leq b_{i+1}$ and $\boldsymbol{R} = \bigcup_{i \in I} [a_i, b_i]$. Such a sequence $\mathbf{P}$ is called a *tagged covering* $\boldsymbol{R}$ *partition sequence*.

Definition 5.3.1

(a) The *partial* (*Riemann*) *hyperintegral* of the function $f(x)$ over the tagged covering $\boldsymbol{R}$ partition sequence $\mathbf{P}$ is equal to

$$\int_{\mathbf{P}} f(x)\, \mathrm{d}x = \mathrm{Hn}(SP_i(f))_{i \in \omega}$$

(b) The partial (Riemann) hyperintegral $\int_{\mathbf{P}} f(x)\, \mathrm{d}x$ is called *partial* (*Riemann*) *integral* of $f(x)$ over $\mathbf{P}$ if its value is a real number.

As $f(x)$ is a total function, the partial sum $SP_i(f)$ is also always defined. Consequently, we have the following simple but basic result.

Theorem 5.3.1 *For any total real function f and any tagged covering **R** partition sequence* $\mathbf{P} = \{P_i; i \in \omega\}$, *the partial hyperintegral of f over the sequence* **P** *always exists.*

Let us study properties of partial hyperintegrals and integrals.

Theorem 5.3.2 *For any tagged covering **R** partition sequence* $\mathbf{P} = \{P_i; i \in \omega\}$, *any real function f and any real number c, we have*

$$\int_{\mathbf{P}} cf(x)\,\mathrm{d}x = c\int_{\mathbf{P}} f(x)\,\mathrm{d}x$$

Proof is similar to the proof of Theorem 5.2.2.

Theorem 5.3.3 *For any tagged covering **R** partition sequence* $\mathbf{P} = \{P_i; i \in \omega\}$, *any two real functions f and g, we have*

$$\int_{\mathbf{P}} (f(x) + g(x))\,\mathrm{d}x = \int_{\mathbf{P}} f(x)\,\mathrm{d}x + \int_{\mathbf{P}} g(x)\,\mathrm{d}x$$

Proof is similar to the proof of Theorem 5.2.3.

Corollary 5.3.1 *For any tagged covering **R** partition sequence* $\mathbf{P} = \{P_i; i \in \omega\}$, *the partial Riemann hyperintegral* $\int_{\mathbf{P}}$ *is a linear hyperfunctional in the space of all real functions.*

Note that if $\int_{\mathbf{P}}$ is always equal to a real number, then it is a linear functional in the space of all real functions.

Theorem 5.3.4 *For any tagged covering **R** partition sequence* $\mathbf{P} = \{P_i; i \in \omega\}$ *and any two real functions f and* g *such that* $f \leq g$, *we have*

$$\int_{\mathbf{P}} f(x)\,\mathrm{d}x \leq \int_{\mathbf{P}} g(x)\,\mathrm{d}x$$

Proof is similar to the proof of Theorem 5.2.4.

Let us take a tagged covering ***R*** partition sequence **R**.

Theorem 5.3.5

(a) *If* $\alpha = \int_{\mathbf{R}} f(x)\,\mathrm{d}x$ *and* α *is either an infinitely increasing or infinite positive or infinite expanding hypernumber, then for any hypernumber* β, *there is a subsequence* **Q** *of* **R** *such that* $\int_{\mathbf{Q}} f(x)\,\mathrm{d}x > \beta$.

(b) *If* $\alpha = \int_{\mathbf{R}} f(x)\,\mathrm{d}x$ *and* α *is either an infinitely decreasing or infinite negative or infinite expanding hypernumber, then for any hypernumber* $\eta \in \boldsymbol{R}_\omega$, *there is a subsequence* **P** *of* **R** *such that* $\int_{\mathbf{P}} f(x)\,\mathrm{d}x < \eta$.

Proof is similar to the proof of Theorem 4.2.1.

Now let us consider the extension to the space ***R*** of a given integral on finite intervals, e.g., Riemann or Lebesgue integral.

Let us consider a cumulative interval cover $R = \{[a_i, b_i]; i = 1, 2, 3, \ldots\}$ of $\boldsymbol{R}$ and assume that a real function f is integrable (for convenience, we take the Riemann integrability) on each interval $[a_i, b_i]$ with $i = 1, 2, 3, \ldots$.

Definition 5.3.2 The *partial* (*Riemann*) *hyperintegral* of the function $f(x)$ over $\boldsymbol{R}$ is equal to

$$\int_R f(x)\,\mathrm{d}x = \mathrm{Hn}(I_i)_{i\in\omega}$$

where $I_i = \int_{a_i}^{b_i} a_i f(x)\,\mathrm{d}x$ with $i = 1, 2, 3, \ldots$.

Definition 5.3.3 A real function f is called *locally integrable* if it is integrable on any finite interval.

The following result shows that it is possible to reduce extension integration to generation integration.

Theorem 5.3.6 *For any locally (Riemann) integrable real function f and any cumulative interval cover R of* $\boldsymbol{R}$, *there is a tagged expanding partition sequence* $\mathbf{P} = \{P_i; i \in \omega\}$ *such that*

$$\int_R f(x)\,\mathrm{d}x = \int_{\mathbf{P}} f(x)\,\mathrm{d}x$$

Proof Let us take a locally (Riemann) integrable real function f and a cumulative interval cover $R = \{[a_i, b_i]; i = 1, 2, 3, \ldots\}$ of $\boldsymbol{R}$.

By definition, $I_i = \int_{a_i}^{b_i} f(x)\,\mathrm{d}x = \mathrm{Hn}(SP_{ij}(f))_{j\in\omega} = \lim_{j\to\infty} SP_{ij}$where partitions P_{ij} of the interval $[a_i, b_i]$ form a converging $\boldsymbol{R}$ tagged interval partition sequence $\mathbf{P}_i$. As this sequence is converging, for each $i = 1, 2, 3, \ldots$, it is possible to find a partition $P_{ij} = P_{ij(i)}$ such that $|I_i - SP_{ij(i)}| < 1/i$.

Let us consider the sequence $\mathbf{P} = \{P_{ij}; i \in \omega\}$. It is a tagged covering $\boldsymbol{R}$ partition sequence and by Definition 5.3.1, we have

$$\int_{\mathbf{P}} f(x)\,\mathrm{d}x = \mathrm{Hn}(SP_i(f))_{i\in\omega}$$

At the same time,

$$\int_R f(x)\,\mathrm{d}x = \mathrm{Hn}(I_i)_{i\in\omega}$$

and by the definition of hypernumbers, $\mathrm{Hn}(SP_i(f))_{i\in\omega} = \mathrm{Hn}(I_i)_{i\in\omega}$.

Theorem is proved.

Let us consider a cumulative interval cover R of $\boldsymbol{R}$.

Theorem 5.3.7

(a) *If* $\alpha = \int_R f(x)\,\mathrm{d}x$ *and* α *is either an infinitely increasing or infinite positive or infinite expanding hypernumber, then for any hypernumber* $\beta \in \boldsymbol{R}_\omega$*, there is a subcover* Q *of* R *such that* $\int_Q f(x)\,\mathrm{d}x > \beta$.

(b) *If* $\alpha = \int_R f(x)\,\mathrm{d}x$ *and* α *is either an infinitely decreasing or infinite negative or infinite expanding hypernumber, then for any hypernumber* $\eta \in \boldsymbol{R}_\omega$*, there is a subcover* P *of* R *such that* $\int_P f(x)\,\mathrm{d}x < \eta$.

Proof is similar to the proof of Theorem 4.2.1.

Extension partial integrals are closely related to improper integrals of real functions. Here we consider only improper integrals over the real line $\boldsymbol{R}$. Other improper integrals are treated in the same way.

Let us consider a locally (Riemann) integrable real function $f(x)$.

Definition 5.3.4 (cf., e.g., Gemignani 1971; Ross 1996) A number I is called the *improper* (*Riemann*) *integral* of f over $\boldsymbol{R}$ if

$$I = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{a\to\infty} I_a^+ + \lim_{c\to\infty} I_c^-$$

where

$$I_a^+ = \int_0^a f(x)\mathrm{d}x \quad \text{and} \quad I_c^- = \int_{-c}^0 f(x)\,\mathrm{d}x$$

Theorem 5.3.8 *The improper integral of a real function* f *over* $\boldsymbol{R}$ *exists if and only if for any two cumulative interval covers* $R = \{[0, c_i]; i = 1, 2, 3, \ldots\}$ *and* $Q = \{[0, d_i]; i = 1, 2, 3, \ldots\}$ *of* $\boldsymbol{R}^+ = [0, \infty)$*, the corresponding partial hyperintegrals of* f *over* $\boldsymbol{R}^+$ *coincide, i.e.,*

$$\int_R f(x)\,\mathrm{d}x = \int_Q f(x)\,\mathrm{d}x$$

and for any two cumulative interval covers $U = \{[0, -u_i]; i = 1, 2, 3, \ldots\}$ *and* $V = \{[0, -v_i]; i = 1, 2, 3, \ldots\}$ *of* $\boldsymbol{R}^- = (-\infty, 0]$*, the corresponding partial hyperintegrals of* f *over* $\boldsymbol{R}^-$ *coincide, i.e.,*

$$\int_U f(x)\,\mathrm{d}x = \int_V f(x)\,\mathrm{d}x$$

Proof Necessity. Let us take some cumulative interval covers $R = \{[0, c_i];\ i = 1, 2, 3, \ldots\}$ and $Q = \{[0, d_i];\ i = 1, 2, 3, \ldots\}$ of $\boldsymbol{R}^+$ and a locally integrable real function f for which the improper integral over $\boldsymbol{R}$ exists and is equal to a real number I. Then the limit $\lim_{a\to\infty} I_a^+$ exists and is equal to some real number C. This is equivalent to the condition that for any sequence of numbers $\{a_i;\ i = 1, 2, 3, \ldots\}$ that converges to infinity, i.e., $\lim_{i\to\infty} a_i = \infty$,

$$\lim_{i\to\infty}\int_0^{a_i} f(x)\,\mathrm{d}x = C$$

In particular, we have

$$\lim_{i\to\infty}\int_0^{c_i} f(x)\,\mathrm{d}x = C$$

and

$$\lim_{i\to\infty}\int_0^{d_i} f(x)\,\mathrm{d}x = C$$

Thus, by Proposition 2.1.3, we have

$$\int_R f(x)\,\mathrm{d}x = \mathrm{Hn}\left(\int_0^{c_i} f(x)\,\mathrm{d}x\right)_{i\in\omega} = \lim_{i=\infty}\int_0^{c_i} f(x)\,\mathrm{d}x = C$$

and

$$\int_Q f(x)\,\mathrm{d}x = \mathrm{Hn}\left(\int_0^{d_i} f(x)\,\mathrm{d}x\right)_{i\in\omega} = \lim_{i\to\infty}\int_0^{d_i} f(x)\,\mathrm{d}x = C$$

Consequently,

$$\int_R f(x)\,\mathrm{d}x = \int_Q f(x)\,\mathrm{d}x$$

It shows that the first part of the condition from the theorem is necessary.

In a similar way, we prove necessity of the second part of the condition from the theorem, obtaining necessity of the whole condition.

Sufficiency. Let us assume that for any two cumulative interval covers $R = \{[0, c_i]; i = 1, 2, 3, \ldots\}$ and $Q = \{[0, d_i]; i = 1, 2, 3, \ldots\}$ of $\boldsymbol{R}^+ = [0, \infty]$ corresponding partial hyperintegrals of f over $\boldsymbol{R}^+$ coincide, and for any two cumulative interval covers $U = \{[0, -u_i]; i = 1, 2, 3, \ldots\}$ and $V = \{[0, -v_i]; i = 1, 2, 3, \ldots\}$ of $\boldsymbol{R}^- = [0, -\infty]$ corresponding partial hyperintegrals of f over $\boldsymbol{R}^-$ coincide. Then by Theorem 5.3.7, all partial hyperintegrals of f over $\boldsymbol{R}^+$ and over $\boldsymbol{R}^-$ are finite.

If for some cumulative interval cover R of $\boldsymbol{R}^+$, $\int_R f(x)\,\mathrm{d}x = \alpha = (a_i)_{i\in\omega}$ is an oscillating hypernumber. Then by Proposition 2.1.7, *Spec* α contains, at least, two numbers r and p. By the definition of spectrum, there is a subsequence $\boldsymbol{b} = (b_i)_{i\in\omega}$ of the sequence $\boldsymbol{a} = (a_i)_{i\in\omega}$ such that $r = \mathrm{Hn}(b_i)_{i\in\omega}$ (cf. Chap. 2). Taking a subcover Q of the cover R that corresponds to the subsequence $\boldsymbol{b}$, i.e., elements of which have the same numbers in R as elements of $\boldsymbol{b}$ in $\boldsymbol{a}$, we see that $\int_Q f(x)\ \mathrm{d}x = r$. Consequently, in this case

$$\int_Q f(x)\ \mathrm{d}x \neq \int_R f(x)\ \mathrm{d}x$$

Thus, we come to the conclusion that all hyperintegrals $\int_R f(x)\ \mathrm{d}x$ are equal to one and the same real number, e.g., to d. Consequently, by Proposition 2.1.3, for any sequence of numbers $\{a_i; i = 1, 2, 3, \ldots\}$ that converges to infinity, i.e., $\lim_{i\to\infty} a_i = \infty$,

$$\lim_{i\to\infty} \int_0^{a_i} f(x)\ \mathrm{d}x = d$$

because the cumulative interval cover $R = \{[0, c_i]; i = 1, 2, 3, \ldots\}$ of $\boldsymbol{R}^+$ is arbitrary. By the definition of the limit of a function (cf. Ross 1996; Burgin 2008a),

$$\lim_{a\to\infty} \int_0^{a} f(x)\ \mathrm{d}x = d$$

In a similar way using the second part of the condition from the theorem, we show that for some real number c, we have

$$\lim_{a\to\infty} \int_{-a}^{0} f(x)\ \mathrm{d}x = c$$

This means that the improper integral of a real function f over $\boldsymbol{R}$ exists and is equal to $c + d$.

Theorem is proved.

This allows us to derive properties of improper integrals from properties of partial hyperintegrals. Namely, Theorems 5.3–5.3.4 and Corollary 5.3.1 imply the following results.

Corollary 5.3.2 *For any real function f and any real number c, we have*

$$\int_{-\infty}^{\infty} cf(x)\ \mathrm{d}x = c\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x$$

Corollary 5.3.3 *For any two real functions f and g, we have*

$$\int_{-\infty}^{\infty} (f(x) + g(x))\ \mathrm{d}x = \int_{-\infty}^{\infty} f(x)\ \mathrm{d}x + \int_{-\infty}^{\infty} g(x)\ \mathrm{d}x$$

Corollary 5.3.4 *For the improper Riemann integral* $\int_{-\infty}^{\infty}$ *is a linear functional in the space of real functions.*

Corollary 5.3.5 *For any two real functions f and g such that* $f \leq g$, *we have*

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \leq \int_{-\infty}^{\infty} g(x)\,\mathrm{d}x$$

In a similar way to the conventional improper integral, it is possible to define symmetric improper integral. Often it is called the *Cauchy principal value*.

Definition 5.3.5 A number I is called the *symmetric improper* (*Riemann*) *integral* of f over $\boldsymbol{R}$ if

$$I = \lim_{a \to \infty} I_a$$

where

$$I_a = \int_{-a}^{a} f(x)\,\mathrm{d}x$$

Theorem 5.3.9 *The symmetric improper integral of a real function f over* $\boldsymbol{R}$ *exists if and only if for any two cumulative interval covers* $U = \{[0, -u_i]; i = 1, 2, 3, \ldots\}$ *and* $V = \{[0, -v_i]; i = 1, 2, 3, \ldots\}$ *of* $\boldsymbol{R}$, *the corresponding partial hyperintegrals of f over* $\boldsymbol{R}^-$ *coincide, i.e.,*

$$\int_R f(x)\,\mathrm{d}x = \int_Q f(x)\,\mathrm{d}x$$

Proof is similar to the proof of Theorem 5.3.8.

To conclude this chapter, it is necessary to remark that using appropriate schemas for building partial hyperintegrals, it is possible to encompass not only conventional integrals of different types (Riemann integrals, gauge integrals and improper integrals) but also upper and lower Darboux integrals (Ross 1996), Kolmogorov integral (Kolmogoroff 1925, 1930), integrals of distributions (Schwartz 1950/1951; Antosik et al, 1973), path integral (Feynman 1948; Feynman and Hibbs 1965), Wiener integral (Wiener 1921; Yeh 1973), stochastic integrals (Ito 1944; Protter 2004) and some others (Swartz and Kurtz 2004). This allows one to derive many properties of all these integrals from properties of partial hyperintegrals.

Chapter 6
Conclusion: New Opportunities

The significant problems we face cannot be solved at the same level of thinking we were at when we created them

(Albert Einstein 1879–1955).

Many topics and results in the theory of hypernumbers and extrafunctions have been left beyond the scope of this little book as its goal is to give a succinct introduction into this rich and multilayered theory. Here we briefly describe some of these topics and results, articulating open problems and directions for further research.

One of these problems is integration in infinite-dimensional spaces in general, and the Feynman path integral in particular. Functional integration is central to quantization techniques in theoretical physics. The algebraic properties of functional integrals are exploited to develop series used to calculate properties in quantum electrodynamics and the standard model. In particular, it is now apparent that the path integral is one of the most useful mathematical tools of contemporary theoretical physics. As Kauffman (2001) writes:

> Almost as a concomitant to quantum field theory was Feynman's discovery that virtually all quantum problems could be formulated in terms of integrals and path integrals that presuppose a summation over configurations of classical states. These integrals led to mathematical and physical problems. On the mathematical side the integrals do not always have an associated measure theory.

Although many constructions have been suggested to formalize the concept of the path integral, the problem of making mathematical sense of these matters of integration and renormalization related to the path integral continues to the present day (Kauffman 2001). Even now no rigorous definition of functional integration and in particular, of path integration, has been given in a way that is applicable to all instances of its application in physics. Another way to say this is that some important problems whose solutions are obtained by heuristic methods involving functional integrals have eluded presentation in terms of the conventional mathematical constructions of functional integration. As the majority of experts think,

M. Burgin, *Hypernumbers and Extrafunctions: Extending the Classical Calculus*, SpringerBriefs in Mathematics, DOI 10.1007/978-1-4419-9875-0_6,

making the procedure of functional integration rigorous poses challenges that are the topic of research in the beginning of the twenty-first century.

There are many approaches to the Feynman path integral (cf. Feynman and Hibbs 1965, Kashiwa 1997, Johnson and Lapidus 2002). Diverse constructions of the path integral are often defined for different classes of functions. Developing a definition that has properties similar to ordinary integration is still considered as a big problem for path integration. Different mathematicians and physicists continue their efforts in building a general theory to make rigorous mathematical sense of Feynman's informal ideas.

One of these approaches is hyperintegration in functional spaces (Burgin 2004, 2008/2009). It is based on the theory of hypernumbers, providing means for a sufficiently broad and rigorous mathematical theory of the path integral, as well as substantiating the original ideas of Feynman. The theory of hypernumbers and extrafunctions is well suited for problems of physics because it emanated from physically directed thinking and was derived by a natural extension of the classical approach to the real and complex number universes (Burgin 2002).

Another important class of problems in contemporary physics involves infinite values that appear from calculations with formulas. As it is known, many mathematical models used in quantum theory, such as gauge theories, entail divergence of analytically calculated properties of physical systems. The simplest example is the theory of a free electron that describes its interaction with photons in such a manner that the energy becomes infinite (in the mathematical model of a free electron). Mathematical investigation of many other physical problems also gives rise to divergent integrals and series, which take, in some sense, infinite values. However, physical measurements give, as the result, only finite values. That is why, many methods of divergence elimination (regularization), i.e., of elimination of infinity, have been elaborated. Nevertheless the majority of them are not well grounded mathematically because they use operations with formal expressions that have neither mathematical nor physical meaning. Moreover, such infinities occur in models in physics that cannot be eliminated by these methods.

In contrast to this, the theory of hypernumbers and extrafunctions allows physicists to consistently operate with all infinite physical quantities, making all divergent integrals and series that appear in the calculations with physical formulas correctly grounded as rigorous mathematical objects. It is interesting that this theory allows making renormalization procedures mathematically rigorous, as well as eliminating such procedures from physics and working with infinite values by redefining mathematical models of measurement procedures, which give, as it is in reality, only finite results.

It is interesting to know that diverging integrals and series are frequently used not only in physics but also in many numerical computations. As an example, we can take asymptotic series. They are used for approximation of functions in numerical computations. The main characteristic of an asymptotic series is that its partial sums diverge to infinity for a fixed value of x. For instance, the infinite Stirling asymptotic series does not converge.

At the same time, we can approximate certain values of x with a precision that increases with the number of terms used. Moreover, taking finite partial sums of that series, we see that the approximation of a function by an asymptotic series becomes more and more accurate when the value of x approaches infinity. This makes the formula rather efficient for approximation purposes. Nevertheless, there will come a certain point in the series (this point depends on what value of x is chosen) in which further terms, instead of giving better and better approximation, will behave differently, thus the infinite partial sum will diverge in the end.

In a similar way, mathematicians use transcendent and other irrational numbers, the exact numerical value of which cannot be represented in the used numerical systems. For instance, it is impossible to write/compute all digits in the decimal representation even of such a simple real number as $\sqrt{2}$. However, mathematicians and scientists successfully use real numbers, both rational and irrational, because they have exact rules to operate with them. Alike, the theory of hypernumbers and extrafunctions gives exact rules to operate with divergent series and integrals, providing sound foundations for asymptotic analysis and opening new possibilities for asymptotic methods, which have particular importance in analysis, as well as in many areas of physics and applied mathematics. For instance, it is possible to consider asymptotic expansions of a function as extrafunctions that belong to a functional neighborhood of a given function. Such functional neighborhoods are used for nonlinear extensions of hypernumbers by Burgin (2005a), which are utilized for defining multiplication of hypernumbers.

There are models in physics that use oscillating integrals, such as the integral $\int_{-\infty}^{\infty} f(x) \sin x \, dx$ (Connor 1990; Connor and Hobbs 2004). For instance, oscillating integrals often describe the scattering of atoms and molecules under short wave-length (or high frequency) conditions or are used in spectroscopic problems, as well as in the theory of water, geophysical, electromagnetic, and acoustic waves. The conventional integration theory does not assign analytical meaning to oscillating integrals and their evaluation is usually done using uniform asymptotic techniques. In contrast to this, the theory of hypernumbers and extrafunctions provides rigorous interpretation for oscillating integrals in a form of oscillating numbers.

One more application of the theory of hypernumbers and extrafunctions are differential equations. As we know from the history of mathematics, extension of rational numbers to real numbers allowed mathematicians to solve much more algebraic equations than it had been possible before. Introduction of complex numbers allowed mathematicians to solve arbitrary algebraic equations. In a similar way, extension of real and complex numbers to real and complex hypernumbers allowed mathematicians to solve much more differential equations than it had been possible before. For instance, it has been proved that many differential equations that do not have solutions even in distributions become solvable when we use extrafunctions (Burgin and Ralston 2004; Burgin 2010). It is necessary to remark that while for linear partial differential equations (PDF) a very advanced version of the Cauchy–Kowalewski theorem was obtained by Burgin and Ralston (2004), in the area of nonlinear differential equations, only first steps have been made (cf. Burgin 2010). Many problems are still open and waiting for mathematicians to solve them.

Probability theory is nowadays a well-developed mathematical field, which forms a foundation for statistics, and an important tool in most sciences, engineering and industry. Probability theory is usually defined as the branch of mathematics concerned with analysis of random phenomena, for which probability is the central concept. However, there is no unique understanding of the term *probability* and there is no single mathematical system that formalizes the term *probability*, but rather a host of such systems. In addition, investigation of relations between probability theory and science demonstrates that it is necessary to extend probability theory and the concept of probability (Döring and Isham 2008; Hellman 1978; Sudbury 1976).

To build a more comprehensive and coherent theory, hyperprobability has been introduced and studied based on the theory of hypernumbers (Burgin and Krinik 2009; Luu 2011). The concept of hyperprobability eliminates many problems that researchers have with the concept of probability. It extends that scope of the probability theory giving a general schema for considering all sequences of events and not only of random events. In addition, hyperprobability provides for a sound justification of the frequency approach to probability, which is the most important for science and engineering applications. In addition, hyperprobabilities demonstrated their efficiency in a study of nonstationary Markov chains (Luu 2011).

Relations between the theory of extrafunctions and nonsmooth analysis (Clark 1983) are studied by Burgin (2001). In particular, it is demonstrated that generalized derivatives in the sense of nonsmooth analysis are particular cases of sequential partial derivatives.

More general classes of hypernumbers are introduced and explored by Burgin (2001, 2005c).

In this book, we consider only hyperintegrals of real functions, which are linear hyperfunctionals. More general hyperfunctionals are studied by Burgin (1991, 2004). For instance, in Burgin (1991), a version of the Hann–Banach theorem for hyperfunctionals is proved. It would be interesting to explore more functionals and hyperfunctionals in spaces of extrafunctions.

Summation of hypernumbers is studied by Burgin (2008b). It is demonstrated that any series of real numbers has a sum in hypernumbers and when a series is summable in the conventional sense its sum coincides with its sum in hypernumbers. It is also proved that if a series R does not converge in the classical sense, while the absolute value of its analytical sum is bounded, then for any hypernumber α, there is a quotient series Q that is a permutation of a quotient series of the initial series R and converges to α. This result essentially extends the classical Riemann theorem for conditionally convergent series.

One can think of the further development of the theory of hypernumbers and extrafunctions in different directions, e.g., such as the development of the theory of operators and operator algebras in infinite dimensional spaces. Operator algebras are frequently used in theories of quantum fields, theory of chaos, and synergetics. Hypernumbers allow one to define hypernorms on topological spaces and algebras (Burgin 1997). With respect to operators, this makes it possible to consider unbounded operators to a full extent and to study hypernormed algebras of such operators. The crucial point in this direction is the definition of multiplication of hypernumbers and extrafunctions.

To conclude, we formulate some interesting open problems in the theory of hypernumbers and extrafunctions.

Problem 1 Given a continuous real function *f*, is its extrafunction extension *F* (cf. Sect. 3.1.1) also continuous?

Problem 2 Given a uniformly continuous in an interval [*a*, *b*] real function *f*, is its extrafunction extension *F* also uniformly continuous?

Problem 3 Under what conditions is a continuously represented norm-based real extrafunction *F* continuous?

It is proved that $\boldsymbol{R}_\omega$ is a Hausdorff topological space, i.e., a T_2-space.

Problem 4 Is $\boldsymbol{R}_\omega$ a $\mathbf{T}_3$-space?

Problem 5 Is $\boldsymbol{R}_\omega$ a $\mathbf{T}_4$-space?

It is proved that $\boldsymbol{R}_\omega$ is a topological vector space.

Problem 6 When is $\mathbf{E}^{\mathbf{F}}_{\omega Q}$ a topological vector space?

Problem 7 Study relations among such constructions as orders of infinity of Du Bois–Reymond (1870/1871) and Hardy (1910), asymptotic numbers in the sense of Christov and Todorov (1974), asymptotic functions in the sense of Oberguggenberger and Todorov (1998), hypernumbers and extrafunctions.

Problem 8 Does validity of the Product Rule for any sequential partial derivative of the product of the given two real functions *f* and *g* at a point *a* implies continuity of these functions *f* and *g* at this point *a*?

Problem 9 Is there a continuous function *f*(*x*) such that for some real number *a* and for any real hypernumber α, there is an A-approximation *R* of *a* such that $\partial/_{\partial R} f(a) = \alpha$?

Summation of hypernumbers is studied by Burgin (2008b). This brings us to the following problem.

Problem 10 Study summation of extrafunctions.

It is proved (in Chap. 5) that there is a real function *f*(*x*) such that for any hypernumber β, there is a partial hyperintegral of *f*(*x*) over the interval [−1, 1] that is equal to β.

Problem 11 Find conditions when the partial hyperintegral $\int_{\mathbf{P}} f(x)\,\mathrm{d}x$ of a function *f*(*x*) over a tagged partition sequence **P** for a given interval [*a*, *b*] can be equal to any real hypernumber.

Problem 12 Apply the theory of hypernumbers and extrafunctions to problems in quantum physics.

Appendix

1 General Concepts and Structures

$N = \{1, 2, 3, \ldots\}$ is the set of all natural numbers;
ω is the sequence of all natural numbers;
∅ is the *empty set*, i.e., the set that has no elements.
$\boldsymbol{R}$ is the set of all real numbers;
$\boldsymbol{R}^+$ is the set of all non-negative real numbers;
$\boldsymbol{R}^{++}$ is the set of all positive real numbers;
$\boldsymbol{C}$ is the set of all complex numbers;

If a is a real number, then $|a|$ or $||a||$ denotes its absolute value or modulus;
If a is a complex number, then $||a||$ or $|a|$ denotes its magnitude or modulus;
If b is a vector from $\boldsymbol{R}^n$, then $||b||$ denotes its modulus;
In general, a sequence of elements a_i is denoted either by $\{a_i(x); i = 1, 2, 3, \ldots\}$ or by $\{a_i(x); i \in \omega\}$ or by $(a_i)_{i\in\omega}$;

$F(\boldsymbol{R})$ is the space of all real functions;
$C(\boldsymbol{R})$ is the space of all continuous real functions;
$F[a,b]$ is the space of all real functions defined in the interval $[a,b]$.
$C[a,b]$ is the space of all continuous real functions defined in the interval $[a,b]$.

If X is a set (class), then $r \in X$ means that r belongs to X or r is a member of X.
If X and Y are sets (classes), then $Y \subseteq X$ means that Y is a *subset* (subclass) of X, i.e., Y is a set such that all elements of Y belong to X, and X is a *superset* of Y. A subset is *proper* if it does coincide with the whole set.
The *union* $Y \bigcup X$ of two sets (classes) Y and X is the set (class) that consists of all elements from Y and from X. The union $Y \bigcup X$ is called disjoint if $Y \cap X = \varnothing$.
The *intersection* $Y \cap X$ of two sets (classes) Y and X is the set (class) that consists of all elements that belong both to Y and to X.
The *union* $\bigcup_{i\in I} X_i$ of sets (classes) X_i is the set (class) that consists of all elements from all sets (classes) X_i, $i \in I$.

M. Burgin, *Hypernumbers and Extrafunctions: Extending the Classical Calculus*, SpringerBriefs in Mathematics, DOI 10.1007/978-1-4419-9875-0,

The *intersection* $\bigcap_{i \in I} X_i$ of sets (classes) X_i is the set (class) that consists of all elements that belong to each set (class) X_i, $i \in I$.

The *difference* $Y \backslash X$ of two sets (classes) Y and X is the set (class) that consists of all elements that belong to Y but does not belong to X.

The *symmetric difference* $Y \Delta X$ of two sets (classes) Y and X is equal to $(Y \backslash X) \bigcup (Y \backslash X)$.

If X is a set, then 2^X is the *power set* of X, which consists of all subsets of X. The *power set* of X is also denoted by PX.

If X and Y are sets (classes), then $X \times Y = \{(x, y); x \in X, y \in Y\}$ is the direct or Cartesian product of X and Y, in other words, $X \times Y$ is the set (class) of all pairs (x, y), in which x belongs to X and y belongs to Y.

Y^X is the set of all mappings from X into Y.

$$X^n = \underbrace{X \times X \times \cdots \times X \times X}_{n}.$$

Elements of the set X^n have the form $(x_1, x_2, \ldots, x_n)$ with all $x_i \in X$ and are called n-tuples, or simply, tuples.

A fundamental structure of mathematics is *function*. However, functions are special kinds of binary relations between two sets.

A *binary relation* T between sets X and Y, also called *correspondence* from X to Y, is a subset of the direct product $X \times Y$. The set X is called the *domain* of $T(X = \text{Dom}(T))$ and Y is called the *codomain* of $T(Y = \text{Codom}(T))$. The *range* of the relation T is $\text{Rg}(T) = \{y; \exists x \in X((x, y) \in T)\}$. The *domain of definition* also called the *definability domain* of the relation T is $\text{DDom}(T) = \{x; \exists y \in Y((x, y) \in T)\}$. If $(x, y) \in T$, then one says that the elements x and y are in relation T, and one also writes $T(x, y)$.

The image $T(x)$ of an element x from X is the set $\{y; (x, y) \in T\}$ and the coimage $f^{-1}(y)$ of an element y from Y is the set $\{x; (x, y) \in T\}$.

Binary relations are also called *multivalued functions* (mappings or maps).

Taking binary relations $T \subseteq X \times Y$ and $R \subseteq Y \times Z$, it is possible to build a new binary relation $RT \subseteq X \times Z$ that is called the (*sequential*) *composition* or *superposition* of binary relations T and R and defined as

$$RT = \{(x, z);\ x \in X,\ z \in Z;\ \text{where } (x, y) \in T \text{ and } (y, z) \in R \text{ for some } y \in Y\}.$$

Sequential composition or superposition of binary relations defines sequential composition or superposition of functions.

A *preorder* (also called *quasiorder*) on a set (class) X is a binary relation Q on X that satisfies the following axioms:

O1. Q is *reflexive*, i.e., xQx for all x from X.
O2. Q is *transitive*, i.e., xQy and yQz imply xQz for all $x, y, z \in X$.

A *partial order* is a preorder that satisfies the following additional axiom:

O3. Q is *antisymmetric*, i.e., xQy and yQx imply $x = y$ for all $x, y \in X$.

A *strict* also called *sharp partial order* is a preorder that is not reflexive, is transitive and satisfies the following additional axiom:

O4. Q is *asymmetric*, i.e., only one relation xQy or yQx is true for all $x, y \in X$.

A *linear* or *total order* is a strict partial order that satisfies the following additional axiom:

O5. We have either xQy or yQx for all $x, y \in X$.

A set (class) X is *well-ordered* if there is a partial order on X such that any its non-empty subset has the least element. Such a partial order is called *well-ordering*. An *equivalence* on a set (class) X is a binary relation Q on X that is reflexive, transitive and satisfies the following additional axiom:

O6. Q is *symmetric*, i.e., xQy implies yQx for all x and y from X.

If we have an equivalence σ on a set X, this set is a disjoint union of classes of the equivalence σ where each class consists of equivalent elements from X and there are no equivalent elements in different classes.

A *tolerance relation* is a binary relation that is reflexive and symmetric.

A *function* (also called a *mapping* or *map* or *total function* or *total mapping* or *everywhere defined function*) f from X to Y is a binary relation between sets X and Y in which there are no elements from X which are corresponded to more than one element from Y and to any element from X, some element from Y is corresponded. At the same time, the function f is also denoted by $Reitz_1 18146$ or by $f(x)$. In the latter formula, x is a variable and not a definite element from X. The *support*, or *carrier*, of a function f is the closure of the set where $f(x) \neq 0$.

Traditionally, the element $f(a)$ is called the *image* of the element a and denotes the value of f on the element a from X. The *coimage* $f^{-1}(y)$ of an element y from Y is the set $\{x; f(x) = y\}$.

There are three basic form of function representation (definition):

1. (The *table representation*) A function f is given as a table or list of pairs (x, y) where the first element x is taken from X, while the second element y is the image $f(x)$ of the first one.
2. (The *analytic representation*) A function f is described by a formula, i.e., a relevant expression in a mathematical language, e.g., $f(x) = \sin\left(e^{x+\cos x}\right)$.
3. (The *algorithmic representation*) A function f is given as an algorithm that computes $f(x)$ given x.

There is a dynamic way of function definition. A (*partial*) *function* f from X to Y is a rule that assigns an element from Y to each element (some elements) from X. When the function f is so defined, then the corresponding binary relation between sets X and Y is called the *graph* of the function f. When X and Y are sets of points in a geometrical space, e.g., their elements are real numbers, the graph of the function f is called the *geometrical graph* of f.

A *partial function* (or *partial mapping*) f from X to Y is a binary relation between sets X and Y in which there are no elements from X which are associated to more than one element from Y. Thus, any function is also a partial function. Sometimes, when the domain of a partial function is not specified, we call it simply a function because any partial function is a total function on its domain.

A *multivalued function* (or *mapping*) f from X to Y is any binary relation between sets X and Y.

$f(x) \equiv a$ means that the function $f(x)$ is equal to a at all points where $f(x)$ is defined.

A function (mapping) f from X to Y is an *injection* if the equality $f(x) = f(y)$ implies the equality $x = y$ for any elements x and y from X, i.e., different elements from X are mapped into different elements from Y.

A function (mapping) f from X to Y is a *projection* also called *surjection* if for any y from Y there is x from X such that $f(x) = y$.

A function (mapping) f from X to Y is a *bijection* if it is both a projection and injection.

A function (mapping) f from X to Y is an *inclusion* if the equality $f(x) = x$ holds for any element x from X.

Two important concepts of mathematics are the domain and range of a function. However, there is some ambiguity for the first of them. Namely, there are two distinct meanings in current mathematical usage for this concept. In the majority of mathematical areas, including the calculus and analysis, the term "domain of f" is used for the set of all values x such that $f(x)$ is defined. However, some mathematicians (in particular, category theorists), consider the domain of a function $f : X \to Y$ to be X, irrespective of whether $f(x)$ is defined for all x in X. To eliminate this ambiguity, we suggest the following terminology consistent with the current practice in mathematics.

If f is a function from X into Y, then the set X is called the *domain* of f (it is denoted by Dom f) and Y is called the *codomain* of T (it is denoted by Codom f). The *range* Rg f of the function f is the set of all elements from Y assigned by f to, at least, one element from X, or formally, $\mathrm{Rg} f = \{y; \exists x \in X(f(x) = y)\}$. The *domain of definition* also called the *definability domain*, DDom f, of the function f is the set of all elements from X that related by f to, at least, one element from Y is or formally, $\mathrm{DDom} f = \{x; \exists y \in Y(f(x) = y)\}$. Thus, for a partial function f, its domain of definition DDom f is the set of all elements for which $f(x)$ is defined.

Taking two mappings (functions) $f : X \to Y$ and $g : Y \to Z$, it is possible to build a new mapping (function) $gf : X \to Z$ that is called the (*sequential*) *composition* or *superposition* of mappings (functions) f and g and defined by the rule $gf(x) = g(f(x))$ for all x from X.

For any set S, $\chi_S(x)$ is its characteristic function, also called *set indicator function*, if $\chi_S(x)$ is equal to 1 when $x \in S$ and is equal to 0 when $x \notin S$, and $\mathrm{C}_S(x)$ is its partial characteristic function if $\mathrm{C}_S(x)$ is equal to 1 when $x \in S$ and is undefined when $x \notin S$.

If $f : X \to Y$ is a function and $Z \subseteq X$, then the restriction $f|_Z$ of f on Z is the function defined only for elements from Z and $f|_Z(z) = f(z)$ for all elements z from Z.

A real function $f : \boldsymbol{R} \to \boldsymbol{R}$ is *bounded* if there is a real number c such that $|f \times (x)| < c$ for all elements x from $\boldsymbol{R}$.

A real function f is called *Lipschitz continuous* if there exists a real number $K \geq 0$, such that for all real numbers x and y, we have

$$|f(x) - f(y)| \leq K|x - y|.$$

A real function f is called a *contraction* or *contraction function* if there exists a real number k, such that $0 \leq k \leq 1$ and for all real numbers x and y, we have

$$|f(x) - f(y)| < k|x - y|$$

i.e., a contraction is Lipschitz continuous for $K < 1$.

A real function f is called *periodic* if there exists a real number $k > 0$, such that for all real numbers x, we have $f(x) = f(x + k)$.

A function f from a partially ordered set X to a partially ordered set X is called *monotone* (*antitone*) if $x \leq y$ implies $f(x) \leq f(y)(f(y) \leq f(x))$ for any elements x and y from X.

If U is a correspondence of a set X to a set Y (a binary relation between X and Y), i.e., $U \subseteq X \times Y$, then $U(x) = \{y \in Y; (x, y) \in U\}$ and $U^{-1}(y) = \{x \in X; (x, y) \in U\}$.

An n-ary relation R in a set X is a subset of the nth power of X, i.e., $R \subseteq X^n$. If $(a_1, a_2, \ldots, a_n) \in R$, then one says that the elements $a_1, a_2, \ldots, a_n$ from X are in relation R.

2 Logical Concepts and Structures

If P and Q are two statements, then P $\rightarrow$ Q means that P implies Q and P $\leftrightarrow$ Q means that P is equivalent to Q.

Logical operations:

– *Negation* is denoted by $\neg$ or by $\sim$
– *Conjunction* also called logical "and" is denoted by $\wedge$ or by & or by$\cdot$
– *Disjunction* also called logical "or" is denoted by $\vee$
– *Implication* is denoted by $\rightarrow$ or by $\Rightarrow$ or by $\supset$
– *Equivalence* is denoted by $\leftrightarrow$ or by $\equiv$ or by $\Leftrightarrow$

The logical symbol $\forall$ is called the *universal quantifier* and means "for any."

The logical symbol $\exists$ is called the *existential quantifier* and means "there exists."

Logical formulas and operations allow mathematicians and logicians to work both with finite and infinite systems. However, to do this in a more constructive form when the number of elements in the system is very big or infinite, mathematicians use the Principle of Induction.

Descriptive principle of induction. Given an infinite (very big) system $R = \{x_1, x_2, \ldots, x_n, \ldots\}$ of elements enumerated by natural numbers and a predicate P

defined for these elements, if it is proved that $P(x_1)$ is true and assuming for an arbitrary number n, that all $P(x_1), P(x_2), \ldots, P(x_{n-1})$ are true, it is also proved that $P(x_n)$ is true, then $P(x)$ is true for all elements from R.

Constructive principle of induction. If an infinite (very big) system $R = \{x_1, x_2, \ldots, x_n, \ldots\}$ of elements enumerated by natural numbers is described by some property represented by a predicate P defined for elements of R, then if there is an algorithm (constructive method) A that builds x_1 and assuming that for an arbitrary number n, all $x_1, x_2, \ldots, x_{n-1}$ are built, A can also build x_n, then can (potentially) build the whole R, i.e., R exists potentially.

3 Topological Concepts and Structures

A *topology* in a set X is a system $\mathrm{O}(X)$ of subsets of X that are called *open subsets* and satisfy the following axioms:

T1. $X \in \mathrm{O}(X)$ and $\varnothing \in \mathrm{O}(X)$.
T2. For all subsets A and B of X, if $A, B \in \mathrm{O}(X)$, then $A \cap B \in \mathrm{O}(X)$.
T3. For all subsets A_i of $X (i \in I)$, if all $A_i \in \mathrm{O}(X)$, then $\bigcup_{i \in I} A_i \in \mathrm{O}(X)$.

A set X with a topology in it is called a *topological space*.

Topology in a set can be also defined by a system of neighborhoods of points from this set. In this case, a set is *open* in this topology if it contains a standard neighborhood of each of its points. For instance, if a is a real number and $t \in \boldsymbol{R}^{++}$, then an open interval $\mathrm{O}_t a = \{x \in \boldsymbol{R}; a - t < x < a + t\}$ is a standard neighborhood of a.

To define a topology, a system of sets has to satisfy the following *neighborhood axioms* (Kuratowski 1966).

NB1. Any neighborhood of a point $x \in X$ contains this point.
NB2. For any two neighborhoods $\mathrm{O}_1 x$ and $\mathrm{O}_2 x$ of a point $x \in X$, there is a neighborhood $\mathrm{O}x$ of x that is a subset of the intersection $\mathrm{O}_1 x \cap \mathrm{O}_2 x$.
NB3. For any neighborhood $\mathrm{O}x$ of a point $x \in X$ and a point $y \in \mathrm{O}x$, there is a neighborhood $\mathrm{O}y$ of y that is a subset of $\mathrm{O}x$.

One more way to define topology in a set is to use the *closure operation* (Kuratowski 1966).

A topology σ in a set X is *stronger* than topology τ in the same set X if any open in the topology τ set is also open in the topology σ.

If X is a subset of a topological space, then $\mathrm{Cl}(X)$ denotes the *closure* of the set X.

A topological space X can satisfy the following axioms (Kelly 1957):

$\mathbf{T}_0$ (the *Kolmogorov Axiom*). $\forall x, y \in X (\exists \mathrm{O}x (y \notin \mathrm{O}x) \vee \exists \mathrm{O}y (x \notin \mathrm{O}y))$.
In other words, for every pair of points a and b there exists an open set U in O such that at least one of the following statements is true: (1) a belongs to U and b does not belong to U, and (2) b belongs to U and a does not belongs to U.
$\mathbf{T}_1$ (the *Alexandroff Axiom*). $\forall x, y \in X \exists \mathrm{O}x \exists \mathrm{O}y (x \notin \mathrm{O}y \ \& \ y \notin \mathrm{O}x)$.

In other words, for every pair of points *a* and *b* there exists an open set U such that U contains *a* but not *b*. To say that a space is T_1 is equivalent to saying that sets consisting of a single point are closed.

$\mathbf{T}_2$ (the *Hausdorff Axiom*). $\forall x, y \in X \exists \mathrm{O}x \exists \mathrm{O}y (\mathrm{O}x \cap \mathrm{O}y = \varnothing)$.

In other words, for every pair of points *a* and *b* there exist disjoint open sets which separately contain *a* and *b*. In this case, open sets *separate* points.

Here O*x*, O*y* are some neighborhoods of *x* and *y*, respectively.

A topological space, which satisfies the axiom $\mathbf{T}_i$, is called a $\mathbf{T}_i$-*space*. Each axiom $\mathbf{T}_{i+1}$ is stronger than axiom $\mathbf{T}_i$. $\mathbf{T}_0$-*spaces* are also called the *Kolmogorov spaces*. $\mathbf{T}_1$-*spaces* are also called the *Fréchet spaces*. $\mathbf{T}_2$-spaces are also called the *Hausdorff spaces* (Kelly 1957).

There are also $\mathbf{T}_3$-*spaces* or *regular spaces*, in which for every point *a* and closed set B there exist disjoint open sets which separately contain *a* and B. That is, points and closed sets are separated. Many authors require that $\mathbf{T}_3$-spaces also be $\mathbf{T}_0$ spaces, since with this added condition, they are also $\mathbf{T}_2$ (Alexandroff 1961).

There are also $\mathbf{T}_4$-*spaces* or *normal spaces*, in which for every pair of closed sets A and B there exist disjoint open sets which separately contain A and B. That is, points and closed sets are separated. Many authors require that $\mathbf{T}_4$-spaces also satisfy Axiom $\mathbf{T}_1$ (Alexandroff 1961).

4 Algebraic Concepts and Structures

A *ring* K is a set with two operations:

Addition: $K \times K \to K$ denoted by $x + y$ where x and y belong to K
Multiplication: $K \times K \to K$ denoted by xy where x and y belong to K

These operations satisfy the following axioms:

1. *Addition is associative*:
   For all x, y, z from K, we have $x + (y + z) = (x + y) + z$.
2. *Addition is commutative*:
   For all x, y from K, we have $x + y = y + x$.
3. *Addition has an identity element*:
   There exists an element 0 from K, called the *zero*, such that $x + 0 = x$ for all x from K.
4. *Addition has inverse element*:
   For any x from K, there exists an element z from K, called the *additive inverse* of x, such that $x + z = 0$.
5. *Multiplication is distributive over addition*:
   For all elements x, y, z from K, we have

$$x(y + z) = xy + xz$$

and

$$(x + y)z = xz + yz$$

Sets $F(\boldsymbol{R})$, $C(\boldsymbol{R})$, $F[a,b]$ and $C[a,b]$ are rings.

A (*left*) *module* M over a ring K if there are two operations:

Addition: $M \times M \to M$ denoted by $x + y$ where x and y belong to M
Multiplication: $K \times M \to M$ denoted by ax where x belongs to M and a belongs to K

These operations satisfy the following axioms:

1. *Addition is associative*:
   For all x, y, z from M, we have $x + (y + z) = (x + y) + z$.
2. *Addition is commutative*:
   For all x, y from M, we have $x + y = y + x$.
3. *Addition has an identity element*:
   There exists an element 0 from M, called the *zero*, such that $x + 0 = x$ for all x from M.
4. *Addition has inverse element*:
   For any x from M, there exists an element z from M, called the *additive inverse* of x, such that $x + z = 0$.
5. *Multiplication is distributive over addition* in M:
   For all elements x from K and y, z from M, we have

$$x(y + z) = xy + xz.$$

6. *Multiplication is distributive over addition* in K:
   For all elements x, from M and y, z from K, we have

$$(z + y)x = zx + yx.$$

In a *right module M*, elements from M are multiplied from the right by elements from M.

Sets $F(\boldsymbol{R})$ and $F[a,b]$ are modules over rings $C(\boldsymbol{R})$ and $C[a,b]$, respectively.

A *linear space* or a *vector space* L over the field $\boldsymbol{R}$ of real numbers has two operations:

Addition: $L \times L \to L$ denoted by $x + y$ where x and y belong to L
Scalar multiplication: $\boldsymbol{R} \times L \to L$ denoted by ax where $a \in \boldsymbol{R}$ and $x \in L$

These operations satisfy the following axioms:

1. Addition is associative:
   For all x, y, z from L, we have $x + (y + z) = (x + y) + z$.
2. Addition is commutative:
   For all x, y from L, we have $x + y = y + x$.

3. Addition has an identity element:
   There exists an element 0 from L, called the *zero vector*, such that $x + 0 = x$ for all x from L.
4. Addition has inverse element:
   For any x from L, there exists an element z from L, called the *additive inverse* of x, such that $x + z = 0$.
5. Scalar multiplication is distributive over addition in L:
   For all elements a from $\boldsymbol{R}$ and vectors y, z from L, we have

$$a(y + z) = ay + az.$$

6. Scalar multiplication is distributive over addition in $\boldsymbol{R}$:
   For all element elements a, b from $\boldsymbol{R}$ and any vector $\boldsymbol{y}$ from L, we have

$$(a + b)y = ay + by.$$

7. Scalar multiplication is compatible with multiplication in $\boldsymbol{R}$:
   For all elements a, b from $\boldsymbol{R}$ and any vector y from L, we have

$$a(by) = (ab)y.$$

8. The identity element 1 from the field $\boldsymbol{R}$ also is an identity element for scalar multiplication:
   For all vectors $\boldsymbol{x}$ from L, we have $1\boldsymbol{x} = \boldsymbol{x}$.

Sets $F(\boldsymbol{R})$, $C(\boldsymbol{R})$, $F[a,b]$ and $C[a,b]$ are vector spaces over the field $\boldsymbol{R}$.

Vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ from L are called *linearly dependent* in L if any there is an equality $\sum_{i=1}^{n} a_i\boldsymbol{x}_i = 0$ where a_i are elements from $\boldsymbol{R}$ and not all of them are equal to 0. When there are no such an equality, vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ are called *linearly independent*.

A system B of linearly independent vectors from L is called a *basis* of L if any element $\boldsymbol{x}$ from L is equal to a sum $\sum_{i=1}^{n} a_i\boldsymbol{x}_i$ where n is some natural number, $\boldsymbol{x}_i$ are elements from B and a_i are elements from $\boldsymbol{R}$.

The number of elements in a basis is called the *dimension* of the space L. It is proved that all bases of the same space have the same number of elements. The number of elements in a basis is called the *dimension* of the space L.

The space $\boldsymbol{R}$ is a one-dimensional vector (linear) space over itself. The space $\boldsymbol{R}^n$ is an n-dimensional vector (linear) space over $\boldsymbol{R}$.

A linear space A over $\boldsymbol{R}$ is called a *linear algebra* over $\boldsymbol{R}$ if a binary operation called multiplication is also defined in A and this operation satisfies the following additional axioms:

1. *Multiplication is distributive over addition* in A:
   For all elements x, y and z from A, we have

$$x(y + z) = xy + xz$$

2. *Multiplication is distributive over addition in* $\boldsymbol{R}$:
For all elements x from M and y, z from $\boldsymbol{R}$, we have

$$(z+y)x = zx + yx$$

Sets $F(\boldsymbol{R})$, $C(\boldsymbol{R})$, $F[a,b]$ and $C[a,b]$ are linear algebras over the field $\boldsymbol{R}$.

Note that any linear algebra is also a ring and thus, it is possible to consider modules over linear algebras.

Let L be a vector (linear) space over $\boldsymbol{R}$, i.e., a real vector (linear) space.

A mapping $q : L \rightarrow \boldsymbol{R}$ is called a *seminorm* if it satisfies the following conditions:

N2. $q(ax) = |a|q(x)$ for any x from L and any number a from $\boldsymbol{R}$,

N3 (the triangle inequality).

$q(x+y) \leq q(x) + q(y)$ for any x and y from L

Lemma A1 *If* $q : L \rightarrow \boldsymbol{R}$ *is a seminorm, then* $q(x) \geq 0$ *for all* $x \in L$.

Proof By N3, we have

$$q(x) + q(-x) \geq q(x + (-x)) = q(0)$$

At the same time, by N2, we have $q(0) = 0 \cdot q(0) = 0$ and $q(-x) = q(x)$. Thus, $q(x) + q(-x) = q(x) + q(x) = 2q(x) \geq 0$ and $q(x) \geq 0$.

Lemma is proved.

Lemma A2 *The triangle inequality implies:*

$q(x) - q(y) \leq q(x+y)$ *for any* x and y from L

Proof is left as an exercise.

A seminorm q is called a *norm* if it satisfies Axioms N2, N3 and one more axiom:

N1. $q(x) = 0$ if and only if $x = 0$ for any x from L

A linear space L with a norm (seminorm) is called a *normed* (*seminormed*) *vector* (*linear*) *space* or simply, a *normed* (*seminormed*) *space*.

The standard notation for the norm of a function f is $||f||$.

The most familiar example of a normed space is the set of real numbers $\boldsymbol{R}$ with the absolute value as a norm.

Let us assume that L is a linear algebra over $\boldsymbol{R}$.

The linear algebra L with a seminorm q is called a *seminormed algebra* if it also satisfies Axioms N2, N3 and one more axiom:

N4. $q(x \cdot y) \leq q(x) \cdot q(y)$ for any x and y from L

The linear algebra L with a norm q is called a *normed algebra* if it satisfies Axioms N1, N2, N3 and one more axiom:

N4a. $q(x \cdot y) = q(x) \cdot q(y)$ for any x and y from L

Let us assume that K is a ring K.

The ring K with a seminorm q is called a *seminormed ring* if it satisfies Axioms N3 and one more axiom:

N4. $q(x \cdot y) \leq q(x) \cdot q(y)$ for any x and y from K

The ring K with a norm q is called a *normed ring* if q also satisfies Axioms N1, N3 and one more axiom:

N4a. $q(x \cdot y) = q(x) \cdot q(y)$ for any x and y from K

Let us assume that L is a module over a seminormed (normed) ring K with a seminorm (norm) p.

The module L with a seminorm (norm) q is called a *seminormed module* if q satisfies Axiom N3 and one more axiom:

N5. $q(u \cdot y) \leq p(u) \cdot q(y)$ for any y from L and any u from K

The module L with a norm q is called a *normed module* if q satisfies Axioms N1, N3 and one more axiom:

N5a. $q(u \cdot y) = p(u) \cdot q(y)$ for any y from L and any u from K

The module (ring or linear space) L with a set Q of seminorms [norms] is called a *Q-seminormed* [*Q-normed*] *module* (*ring* or *linear space*) if it is a seminormed [normed] module (ring or linear space, respectively) for each $q \in Q$.

5 Notations from the Theory of Hypernumbers and Extrafunctions

$\boldsymbol{R}^{\omega}$ is the set of all sequences of real numbers.

If X and Y are topological spaces, then $F(X, Y)$ is the set of all and $C(X, Y)$ is the set of all continuous mappings from X into Y.

If $a = (a_i)_{i \in \omega}$ is a sequence of real numbers, then $\alpha = \mathrm{Hn}(a_i)_{i \in \omega}$ is the real hypernumber determined by $\boldsymbol{a}$.

$\boldsymbol{R}_{\omega}$ is the set of all real hypernumbers.

$\boldsymbol{R}_{\omega}^{+}$ is the set of all real hypernumbers that are larger than or equal to zero.

$F(\boldsymbol{R}_{\omega}, \boldsymbol{R}_{\omega})$ is the set of all (general) real pointwise extrafunctions.

$C(\boldsymbol{R}_{\omega}, \boldsymbol{R}_{\omega})$ of all continuously represented (general) real pointwise extrafunctions.

If $\{f_i; i \in \omega\}$ is a sequence of real functions, then $f = \mathrm{Ep}\{f_i; i \in \omega\}$ is the real restricted pointwise extrafunction determined by the sequence $\{f_i; i \in \omega\}$.

$F(\boldsymbol{R}, \boldsymbol{R}_{\omega})$ is the set of all restricted real pointwise extrafunctions.

$C(\boldsymbol{R}, \boldsymbol{R}_{\omega})$ of all continuously represented restricted complex pointwise extrafunctions.

If $\mathbf{F}$ is a class of real functions, then $\mathbf{E}_{\omega Q}^{\mathbf{F}}$ is the set of all represented in $\mathbf{F}$ Q-based real extrafunctions.

If $\{f_i; i \in \omega\}$ is a sequence of real functions and $\mathbf{F}$ is a class of real functions, then $f = \mathrm{EF}_Q(f_i)_{i \in \omega}$ is the real represented in $\mathbf{F}$ Q-based real extrafunction determined by the sequence $\{f_i; i \in \omega\}$;

$\mathbf{E}_{\omega Q_{\mathrm{pt}}}^{\mathbf{F}}$ is the set of all represented in $\mathbf{F}$ Q_{pt}-based real extrafunctions.

$\mathbf{E}_{\omega Q_{\mathrm{comp}}}^{\mathbf{F}}$ is the set of all represented in $\mathbf{F}$ Q_{comp}-based real extrafunctions.

$\mathbf{E}_{\omega Q_{\mathrm{cp}}}^{\mathbf{F}}$ is the set of all represented in $\mathbf{F}$ Q_{cp}-based real extrafunctions.

$\mathbf{D}_{\mathbf{K}\omega}$ is the class of all real extended distributions with respect to $\mathbf{K}$. It is also denoted by $\mathbf{E}^{\mathbf{F}}_{\omega Q_{\mathbf{K}}}$.

If $\{f_i; i \in \omega\}$ is a sequence of real functions, then $f = \mathrm{Ec}\{f_i; i \in \omega\}$ is the real compactwise extrafunction determined by the sequence $\{f_i; i \in \omega\}$;

Comp $(\boldsymbol{R}, \boldsymbol{R}_\omega)$ is the set of all real compactwise extrafunctions.

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